Parameterized Regular Expressions and Their Languages

Pablo Barceló
Univ. de Chile

Leonid Libkin
U. of Edinburgh

Juan Reutter
U. of Edinburgh

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- Σ: a finite alphabet
- $\mathcal{V}$: a countably infinite set of variables $x, y, z, \ldots,$

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$(0x)^*1(xy)^*$ and $(0|1)^*xy(0|1)^*$ are PREs over $\{0, 1\}$.

Language of PREs?

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We want PREs to define languages over Σ.

How to interpret variables in PREs

For now, variables are interpreted as symbols from Σ.

Given a PRE e over Σ that uses variables $\mathcal{W} \subset \mathcal{V}$:

- A valuation for e is a mapping $\nu : \mathcal{W} \to \Sigma$.

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Example:

$$\begin{array}{rcl} e & = & (0x)^*1(xy)^* \\ \nu(e) & = & (00)^*1(01)^* \end{array} \qquad \nu : \; x \mapsto 0, \;\; y \mapsto 1$$

Semantics for PREs over Σ: Two alternatives

Let e be a PRE over Σ. Then

- $\mathcal{L}_\diamond(e) := \bigcup\{\mathcal{L}(\nu(e)) \mid \nu \text{ is a valuation for } e\}$ (possibility)

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- 00101 is in $\mathcal{L}_\diamond(e)$.

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- 10011 is in $\mathcal{L}_{\Box}(e)$.
- No word of length ≤ 4 is in $\mathcal{L}_{\Box}(e)$

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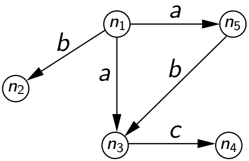
Finite unions or intersections of regular languages:

$\mathcal{L}_\Diamond(e)$ and $\mathcal{L}_\Box(e)$ are regular languages

Applications of PREs: Graph databases

Graph DBs:

- Applications: RDF, SNs, Scientific data, etc.
- Model: Edge-labeled directed graphs (that is: NFAs).



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As it is usual, some data in the graph DB may be missing [Barceló. et al. 2011,Calvanese et al. 2011].

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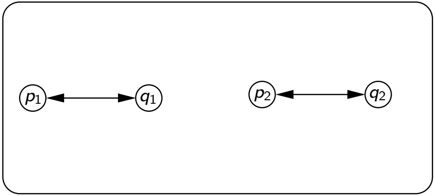

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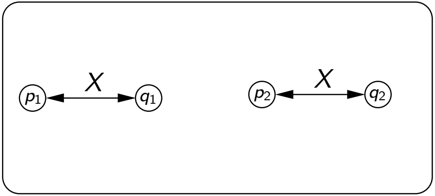

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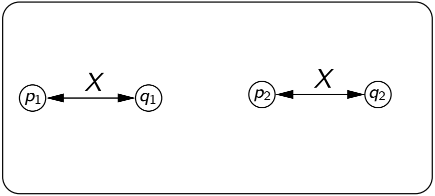

Incomplete graph DBs are graph DBs with edges labeled in $\mathcal{V}$.

- They can be represented as NFAs over $\Sigma \cup \mathcal{V}$.
- Equivalently, as PREs over Σ.

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Standard semantics for incomplete DBs: Certain answers.

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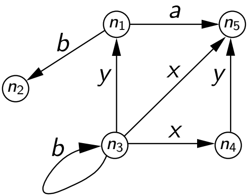
How to use PREs to compute certain answers over graph DBs?

PRE's for querying incomplete graph DB's



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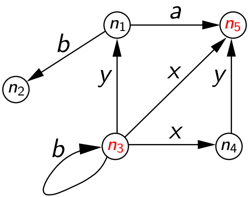
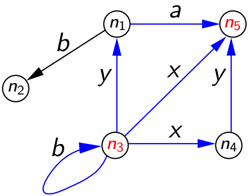


- Paths from n_3 to n_5.

PRE's for querying incomplete graph DB's



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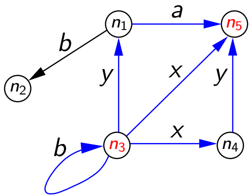
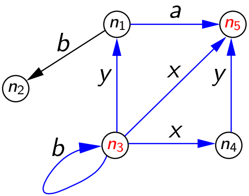
- $e = b^*ya|\ b^*x|\ b^*xy$.

PRE's for querying incomplete graph DB's



- Paths from n_3 to n_5.
- $e = b^* ya|\ b^* x|\ b^* xy$.
- We can be certain about a word $w \in \Sigma^*$ labeling a path from n_3 to n_5 in G iff $w \in \mathcal{L}_{\Box}(e)$.

PRE's for querying incomplete graph DB's



The certainty semantics is essential for computing certain answers over incomplete graph DBs.

Applications of PREs: Program analysis

PREs naturally arise in program analysis [Liu & Stoller 2004, de Moor at al. 2003].

- Alphabet: Operations on variables; e.g. **def**, **use**, **open**, etc.
- Variables: Program variables, pointers, files, etc.

PREs are used in this setting to specify undesired behavior.

Example: The undesired property "A variable is used without being defined" can be expressed as follows:

$$(\neg \mathbf{def}(x))^* \mathbf{use}(x).$$

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These expressions are evaluated over graphs that serve as an abstraction of the program behavior.

Applications of PREs: Program analysis

PREs specify undesired behavior: Assignments of the variables that "satisfy" the PRE represent *bugs* of the program.

In the program analysis context the possibility semantics is essential for finding where the program fails a specification.

We study basic computational problems of PREs

Despite its importance, basic computational problems associated with PREs have not been addressed.

In this paper: Study standard language-theoretical problems for PREs divided as follows:

- Decision problems: Emptiness, universality, containment and membership.
- Computational problems: Minimal-size NFAs representing $\mathcal{L}_\Box(e)$ and $\mathcal{L}_\Diamond(e)$.

- Upper bound techniques
- Decision problems
- Computational problems
- Extending the semantics
- Future work

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NFAs for $\mathcal{L}_\Diamond(e)$ and $\mathcal{L}_\Box(e)$

- Exponentially many valuations: $|\Sigma|^{(\#\text{ of variables})}$.
- Taking the union gives an exponential NFA for $\mathcal{L}_\Diamond(e)$.
- Taking the intersection gives a doubly-exponential NFA for $\mathcal{L}_\Box(e)$.

We shall see that these are tight bounds...

- Upper bound techniques
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In order to do a finer analysis we study two restrictions of PREs:

- Simple: No repetition of variables; e.g. $e = (0|1)^*xy(0|1)^*$.
- Star-height 0: No Kleene-star: i.e. finite languages.

Decision problems: Nonemptiness

- $\textsc{Nonemptiness}_{\Diamond}$: $\mathcal{L}_{\Diamond}(e) \neq \emptyset$?

- $\textsc{Nonemptiness}_{\Box}$: $\mathcal{L}_{\Box}(e) \neq \emptyset$?

Decision problems: Nonemptiness

- $\textsc{Nonemptiness}_{\diamond}$: $\mathcal{L}_{\diamond}(e) \neq \emptyset$?

  Not different from the case without variables:

  $$(0x)^*1^*(xy)^*$$

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- $\textsc{Nonemptiness}_{\Box}$: $\mathcal{L}_{\Box}(e) \neq \emptyset$?

Theorem

$\textsc{Nonemptiness}_{\Box}$ is $\textsc{Expspace}$-complete.

1. Remains $\textsc{Expspace}$-hard even over the class of simple expressions.
2. For PREs of star-height 0: $\textsc{Nonemptiness}_{\Box}$ is Σ_2^P-complete.

PREs and succinct intersection

Main tool for EXPSPACE-hardness:

Given PRE's $e_1, \ldots, e_n$ we can construct in polynomial time a PRE e' such that

$$\mathcal{L}_\Box(e') \text{ is empty iff } \mathcal{L}_\Box(e_1) \cap \cdots \cap \mathcal{L}_\Box(e_n) \text{ is empty.}$$

PREs and succinct intersection

Main tool for EXPSPACE-hardness:

Given PRE's $e_1, \ldots, e_n$ we can construct in polynomial time a PRE e' such that

$$\mathcal{L}_\square(e') \text{ is empty iff } \mathcal{L}_\square(e_1) \cap \cdots \cap \mathcal{L}_\square(e_n) \text{ is empty.}$$

Gives us PSPACE-hardness for NONEMPTINESS$_\square$, since regular expressions are PRE's

Minimal size of words in $\mathcal{L}_{\square}(e)$

Corollary ($\textsc{Nonemptiness}_{\square}$)

There exists a sequence of parameterized regular expressions $\{e_n\}_{n \in \mathbb{N}}$ such that:

1. *Each e_n is of size polynomial in n.*
2. *Every word in the language $\mathcal{L}_{\square}(e_n)$ has size at least 2^{2^n}.*

Minimal size of words in $\mathcal{L}_\square(e)$: exponential bound

Consider PREs of the form:

$$(0 \mid 1)^* \, x_1 \cdot x_2 \cdots x_n \, (0 \mid 1)^* \qquad (n \geq 1).$$

Minimal size of words in $\mathcal{L}_{\square}(e)$: exponential bound

Consider PREs of the form:

$$(0 \,|\, 1)^* \, x_1 \cdot x_2 \cdots x_n \, (0 \,|\, 1)^* \qquad (n \geq 1).$$

- If $w \in \mathcal{L}_{\square}(e)$, then it contains as a subword each $w' \in \{0, 1\}^n$.

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- If $w \in \mathcal{L}_{\square}(e)$, then it contains as a subword each $w' \in \{0, 1\}^n$.

de Bruijn sequences of order n, which are of size $\geq 2^n$.

Decision problems: Universality

$\textsc{Universality}_\circ$: Is $\mathcal{L}_\circ(e) = \Sigma^*$?

As opposed to nonemptiness, universality is more difficult for the $\Diamond$-semantics than for the $\Box$-semantics:

- $\textsc{Universality}_\Box$ is $\textsc{Pspace}$-complete.
- $\textsc{Universality}_\Diamond$ is $\textsc{Expspace}$-complete.
  - It remains $\textsc{Expspace}$-complete even over the class of simple expressions.

Decision problems: Containment

$\textsc{Containment}_\circ$: Is $\mathcal{L}_\circ(e_1) \subseteq \mathcal{L}_\circ(e_2)$?

We can reduce from other problems, since:

- L is empty iff $L \subseteq \emptyset$

Decision problems: Containment

$\textsc{Containment}_\circ$: Is $\mathcal{L}_\circ(e_1) \subseteq \mathcal{L}_\circ(e_2)$?

We can reduce from other problems, since:

- L is empty iff $L \subseteq \emptyset$
- L is Σ^* iff $\Sigma^* \subseteq L$.

Thus,

$\textsc{Containment}_\Box$ and $\textsc{Containment}_\Diamond$ are $\textsc{Expspace}$-complete.

- Even if restricted to simple expressions.

Decision problems: Membership

Is w in $\mathcal{L}_\Diamond(e)$ or $\mathcal{L}_\Box(e)$?

Guess a valuation ν:

- $w \in \mathcal{L}(\nu(e))$ (possibility)
- $w \notin \mathcal{L}(\nu(e))$ (certainty)

Gives us NP and coNP bounds

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Gives us NP and coNP bounds (tight).

Theorem

- $\textsc{Membership}_\Diamond$ is NP-complete.
- $\textsc{Membership}_\Box$ is coNP-complete.

Decision problems: Membership

We can do a finer analysis:

Proposition

- The complexity of $\textsc{Membership}_{\Diamond}$ is as follows:
  1. Simple expressions: NP-complete.
  2. Star-height 0 expressions: NP-complete.
  3. Simple and star-height 0 expressions: $\textsc{Ptime}$.
- The complexity of $\textsc{Membership}_{\Box}$ is as follows:
  1. Simple expressions: $\textsc{co}\mathrm{NP}$-complete.
  2. Star-height 0 expressions: $\textsc{co}\mathrm{NP}$-complete.
  3. Simple and star-height 0 expressions: $\textsc{Ptime}$.

Also in the paper:

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These problems are motivated by the application of PREs

- Upper bound techniques
- Decision problems
- Computational problems
- Extending the semantics
- Future work

Computational problems

What is the size of the minimal NFA $\mathcal{A}$ such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}_\diamond(e)$ or $\mathcal{L}(\mathcal{A}) = \mathcal{L}_\Box(e)$?

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Theorem

The sizes of minimal NFAs are:

- necessarily double-exponential for $\mathcal{L}_\Box$
- necessarily exponential for $\mathcal{L}_\diamond$.

Proof sketch of minimal size NFA for $\mathcal{L}_\Box$

We use the following result by Glaister and Shallit:

If L is a regular language, and there exists a set of pairs

$$P = \{(u_i, v_i) \mid 1 \leq i \leq m\} \subseteq \Sigma^* \times \Sigma^*, \text{ such that}$$

1. $u_i v_i \in L$,
2. $u_j v_i \notin L$ for $i \neq j$,

then every NFA accepting L has at least m states.

Proof skecth of minimal size NFA for $\mathcal{L}_\square$

Consider the following family of PREs:

$$e_n = ((0 \mid 1)^{n+1})^* \cdot x_1 \cdots x_n \cdot x_{n+1} \cdot ((0 \mid 1)^{n+1})^* \qquad (n \geq 1)$$

- Each e_n is of linear size on n.

We shall construct a Fooling Set for $\mathcal{L}_\square(e_n)$.

Proof sketch of minimal size NFA for $\mathcal{L}_\Box$

Given a set $S \subset \{0,1\}^{n+1}$ of size 2^n:

- w_S is the concatenation in lexicographical order of all words in S; and
- $w_{\bar{S},n}$ is the concatenation in lexicographical order of all words in $\{0,1\}^{n+1}$ that are not in S.

Proof skecth of minimal size NFA for $\mathcal{L}_\square$

We define:

$$P_n \;:=\; \{(w_S, w_{\bar{S},n}) \mid S \subset \{0,1\}^{n+1} \text{ and } |S| = 2^n\},$$

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$$P_n \;:=\; \{(w_S, w_{\bar{S},n}) \mid S \subset \{0,1\}^{n+1} \text{ and } |S| = 2^n\},$$

1. There are $\binom{2^{n+1}}{2^n} \geq 2^{2^n}$ different subsets of $\{0,1\}^{n+1}$ of size 2^n, and thus $|P_n| \geq 2^{2^n}$.
2. $(w_S, w_{\bar{S},n})$ belongs to $\mathcal{L}_\Box(e_n)$, but
3. $(w_{S_1}, w_{\bar{S}_2,n})$ are not in $\mathcal{L}_\Box(e_n)$, for distinct S_1 and S_2.

- Upper bound techniques
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Extending Semantics

We can extend semantics and allow replacement of variables by words that belong to some regular language.

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We can extend semantics and allow replacement of variables by words that belong to some regular language.

- $\Diamond$-semantics: Easily becomes non-regular (e.g. xx = squared words). Regular for finite languages.
- $\Box$-semantics: Keeps being regular. Same complexity bounds apply.

- Upper bound techniques
- Decision problems
- Computational problems
- Extending the semantics
- Future work

Future work

Closure properties:

- The minimal NFA $\mathcal{A}$ that accepts $\mathcal{L}_{\Box}(e_1) \cap \mathcal{L}_{\Box}(e_2)$ is necessarily of double-exponential size.

Future work

Closure properties:

- The minimal NFA $\mathcal{A}$ that accepts $\mathcal{L}_{\square}(e_1) \cap \mathcal{L}_{\square}(e_2)$ is necessarily of double-exponential size.
- Perhaps it is possible to construct in polynomial time a PRE e such that $\mathcal{L}_{\square}(e) = \mathcal{L}_{\square}(e_1) \cap \mathcal{L}_{\square}(e_2)$.