Expressiveness of CONSTRUCT Queries in SPARQL

Egor V. Kostylev[1], Juan L. Reutter[2], and Martín Ugarte[2]

1 **University of Oxford**
egor.kostylev@cs.ox.ac.uk
2 **PUC Chile**
jreutter@ing.puc.cl, martinugarte@puc.cl

Abstract

SPARQL has become the most popular language for querying RDF datasets, the standard data model for representing information in the Web. This query language has received a good deal of attention in the last few years: two versions of W3C standards have been issued, several SPARQL query engines have been deployed, and important theoretical foundations have been laid. However, many fundamental aspects of SPARQL queries are not yet fully understood. To this end, it is crucial to understand the correspondence between SPARQL and well-developed frameworks like relational algebra or first order logic. But one of the main obstacles on the way to such understanding is the fact that the well-studied fragments of SPARQL do not produce RDF as output.

In this paper we embark on the study of SPARQL queries with results in CONSTRUCT form, that is, queries which output RDF graphs. This form takes rightful place in the standards and implementations, but, contrary to other forms, has not yet attracted a worth-while theoretical research. Under this framework we are able to establish a strong connection between SPARQL and well-known logical and database formalisms. In particular, the general language can be restated as a data transformation setting, the fragment which does not allow for blank nodes in output templates corresponds to first order queries, and its well-designed sub-fragment corresponds to positive first order queries. This correspondence allows us to conclude that the general language is not composable, but the identified fragments are. Finally, we enrich the language with a recursion operator, and establish fundamental properties of this extension.

Digital Object Identifier 10.4230/LIPIcs.ICDT.2015.1

1 Introduction

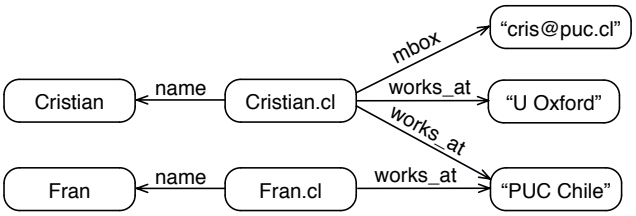
The Resource Description Framework (RDF) [24] is the World Wide Web consortium (W3C) standard for representing linked data on the Web. Intuitively, an RDF graph is set of triples of internationalized resource identifiers (IRIs), where the first and last of them represent entity resources, and the middle one relates these resources, just as is it done with graph databases [3].

SPARQL is a language for querying RDF datasets. First introduced in [33], in 2008 SPARQL was officially made the recommended language to query RDF data by W3C [32], and nowadays this language is recognised as one of the key standards of the Semantic Web initiative. A recent version SPARQL 1.1 of the standard was issued in 2013 [38], and currently there are several SPARQL engines available to industry (e.g., [11, 17, 36]).

The theoretical foundations of SPARQL were laid by Pérez et al. in their seminal work [27], and a body of research has followed, covering a variety of issues such as complexity of query evaluation [5, 23, 29, 35], query optimisation [8, 9, 21, 30], federation [7], expressive power [2, 31], and provenance tracking [14, 16]. The impact of these studies in the

18th International Conference on Database Theory (ICDT'15).
Editors: Marcelo Arenas and Martin Ugarte; pp. 1–25

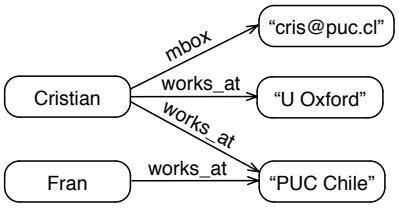
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



(a)

?n	?w	?e
Fran	PUC Chile	
Cristian	U Oxford	cris@puc.cl
Cristian	PUC Chile	cris@puc.cl

(b)



(c)

Figure 1 RDF graph G_{ex} **(a)**; answer of $\boldsymbol{q}_{\text{sel}}$ over G_{ex} is set of mappings **(b)**; answer of $\boldsymbol{q}_{\text{cons}}$ over G_{ex} is RDF graph **(c)**.

Semantic Web community has been astonishing, including influence to the definition of the SPARQL standards.

However, despite the key importance of SPARQL, the fundamental aspects of this language are still not fully understood. Compared to the knowledge on other query languages such as SQL, Datalog or even XPath, very little is known about SPARQL queries. To this end, it is of particular importance to understand the correspondence between SPARQL and other well-developed formalisms such as first order logic or relational algebra. But one of the main obstacles on the way to such understanding is the fact that the queries from well-studied fragments of SPARQL produce not RDF graphs as answers, but sets of mappings (partial evaluations), which are quite different form for representing data.

▶ **Example 1.** As a classical example of SPARQL, let us consider the following query $\boldsymbol{q}_{\text{sel}}$:[1]

```
SELECT ?n, ?w, ?e
WHERE (
((?p, name, ?n) AND (?p, works_at, ?w))
         OPT   (?p, mbox, ?e)).
```

This query is intended to extract all names of people for which a working place is known, together with their affiliations, and, optionally, append their emails, provided the RDF graph contains this information. Thus, when evaluated on the RDF graph G_{ex} from Figure 1(a), it gives as result a set of partial mappings from the variables of $\boldsymbol{q}_{\text{sel}}$ to IRIs in the RDF graph, as depicted in Figure 1(b), where each row represents a mapping.

Returning mappings instead of tuples might appear just as a slight difference between SPARQL and other query languages such as SQL. However, it is known to lead to several complications (see, e.g., [27, 31]). For example, when studying the expressive power of SPARQL in [2, 31], the authors need to use some rather technical machinery to be able to

[1] In this paper we follow the adopted SPARQL syntax of [27], in particular, we shorten `OPTIONAL` to `OPT`.

even compare SPARQL with relational query languages. The result is that, even if we now know that the SELECT fragment of SPARQL is equivalent in expressive power to relational algebra, this is shown using proofs that are much more complicated than other similar results in database theory, and it has been difficult to build upon this proofs to produce new results.

There are also practical consequences: while recursive queries have been part of SQL for more than twenty years, we are still left without a comprehensive operator to define recursive queries in SPARQL (in fact, the recent 1.1 version of SPARQL does include the property paths primitive [38], but only as an additional feature, which is very restrictive in expressing important recursive queries [22]).

However, this complication is relevant only to the SELECT queries of SPARQL, which has been considered in the theoretical literature almost exclusively. Alas, this is not the only form of the results in SPARQL, and there is a result form which outputs RDF graphs, namely the CONSTRUCT result form. The following example illustrates how a user can specify a query with CONSTRUCT result form, that is, a query that outputs an RDF graph.

▶ **Example 2.** Let q_{cons} be the following SPARQL query with CONSTRUCT result form:

```
CONSTRUCT {(?n, works_at, ?w), (?n, mbox, ?e)}
WHERE (
((?p, name, ?n) AND (?p, works_at, ?w))
         OPT   (?p, mbox, ?e)).
```

Note that this query has the same WHERE clause as q_{sel}, but the form of the output is different. The RDF graph resulting from the evaluation of this query over the dataset G_{ex} is depicted in Figure 1(c).

CONSTRUCT queries in SPARQL shape the class of effective queries whose inputs and answers are RDF graphs, so it is conceivable that much more insight can be obtained by comparing them to well-established query languages. But rather surprisingly, despite being an important part of the SPARQL standard, these queries have received almost no theoretical attention, comparing to the queries in SELECT form. This can be partially explained by the fact, that, as the examples above suggest, the difference between these classes of queries might seem negligible. However, as we will see in this paper, this resemblance is often deceptive, and in many cases the properties of these queries are different. For example, CONSTRUCT queries allow for blank nodes in templates, specifying the answer triples, which is a feature unavailable in SELECT queries. Trying to fill this gap, we conduct a thorough study of SPARQL queries of the CONSTRUCT form. We concentrate on the AND-UNION-OPT-FILTER fragment of SPARQL, which is the core of this language [27].

The first question studied in the paper is the expressive power of CONSTRUCT queries. In particular, we show that if blank nodes are not allowed in the templates, then this language is equivalent in expressive power to first order logic. If underlying graph patterns are enforced to be well-designed, that is, to belong to a known fragment with intuitive meaning and good properties ([27]), then the language essentially corresponds to positive first order logic. If, in turn, the blank nodes in templates are allowed, then we establish the equivalence of this full language to a certain data exchange setting.

This expressivity results lead to important conclusions on composability of these languages. In particular, the fragments without blank nodes are composable, that is, the composition of two queries can be always expressed by another query, but if blank nodes are allowed, then this important property is lost.

We also obtain results on computational complexity of evaluation of such queries: for the blank-free language it is the same as for SELECT queries (PSPACE-complete), but for the

well-designed sublanguage there is a surprising difference—it is Σ_2^p-complete for the SELECT case ([21]), but drops to NP-complete in CONSTRUCT case.

Finally, the properties of CONSTRUCT queries allow us to develop an extension of SPARQL with a SQL form of recursion, which unifies several formalisms for querying RDF data, such as SPARQL 1.1 property paths [38], c-query answering over OWL 2 RL entailment regime [15, 19], navigational SPARQL [28], GraphLog [10], and TriAL [22]. For this extension, we also find an expressively equivalent conventional language, namely, a fragment of Datalog.

Due to the space limitations, only ideas of most important proofs are exposed in the main body of this paper, but complete proofs are given in the appendix.

2 Preliminaries

RDF Graphs and Datasets

An RDF graph is a labeled graph where nodes can be edge labels by themselves, and an RDF dataset is a collection of named RDF graphs.

Formally, let $\mathbf{I}$ and $\mathbf{B}$ be infinite pairwise disjoint sets of *IRIs* and *blank nodes*,[2] correspondingly, and $\mathbf{T} = \mathbf{I} \cup \mathbf{B}$ be the set of *terms*. Then an *RDF triple* is a tuple (s, p, o) from $\mathbf{T} \times \mathbf{I} \times \mathbf{T}$, where s is called *subject*, p predicate, and o object. An *RDF graph* is a finite set of RDF triples, and an *RDF dataset* is a set $\{G_0, \langle u_1, G_1\rangle, \ldots, \langle u_n, G_n\rangle\}$, where $G_0, \ldots, G_n$ are RDF graphs and $u_1, \ldots, u_n$ are distinct IRIs, such that the graphs G_i use pairwise disjoint sets of blank nodes. The graph G_0 is called *default graph*, and $G_1, \ldots, G_n$ are called *named graphs* with *names* $u_1, \ldots, u_n$, correspondingly. For a dataset D and IRI u we define $\mathsf{gr}_D(u) = G$ if $\langle u, G\rangle \in D$ and $\mathsf{gr}_D(u) = \emptyset$ otherwise. We also use $\mathcal{G}$ and $\mathcal{D}$ to denote the sets of all RDF graphs and datasets, correspondingly, as well as $\mathsf{blank}(S)$ to denote the set of blank nodes appearing in S, which can be a triple, a graph, etc.

SPARQL Syntax

SPARQL is the standard pattern-matching language for querying RDF datasets.

Formally, let $\mathbf{V}$ be an infinite set $\{?x, ?y, \ldots\}$ of *variables*, disjoint from $\mathbf{T}$. Similarly to $\mathsf{blank}(S)$, $\mathsf{var}(S)$ denotes the set of variables appearing in S.

SPARQL *graph patterns* are recursively defined as follows:

1. a triple in $(\mathbf{I} \cup \mathbf{V}) \times (\mathbf{I} \cup \mathbf{V}) \times (\mathbf{I} \cup \mathbf{V})$ is a graph pattern, called a *triple pattern*;
2. if P_1 and P_2 are graph patterns, then $(P_1 \,\mathsf{AND}\, P_2)$, $(P_1 \,\mathsf{OPT}\, P_2)$, and $(P_1 \,\mathsf{UNION}\, P_2)$ are graph patterns, called AND-, OPT-, and UNION-*patterns*, correspondingly;
3. if P is a graph pattern and $g \in \mathbf{I} \cup \mathbf{V}$ then $(g \,\mathsf{GRAPH}\, P)$ is a graph pattern, called GRAPH-*pattern*;
4. if P is a graph pattern and R is a filter condition, then $(P \,\mathsf{FILTER}\, R)$ is a graph pattern, called FILTER-*pattern*, where SPARQL *filter conditions* are constraints of the form:
   - $?x = u$, $?x = ?y$, $isBlank(?x)$ or $bound(?x)$ for $?x, ?y \in \mathbf{V}$ and $u \in \mathbf{I}$ (called *atomic constraints*[3]),
   - $\neg R$, $R_1 \wedge R_2$, or $R_1 \vee R_2$ for filter conditions R, R_1 and R_2.

[2] For the sake of simplicity we do not consider literals, but all the results in this paper hold if we introduce them explicitly.

[3] We use a simplified list of SPARQL atomic constraints, for the complete one see [38].

The fragment of SPARQL graph patterns has drawn most of the attention in the Semantic Web community. In this paper we concentrate on another class of queries, formalized next.

A SPARQL *query with construct result form*, or *c-query* for short, is an expression

$$\mathsf{CONSTRUCT}\ H\ \mathsf{WHERE}\ P,$$

where H is a set of triples from $(\mathbf{T}\cup\mathbf{V})\times(\mathbf{I}\cup\mathbf{V})\times(\mathbf{T}\cup\mathbf{V})$, called *template*, and P is a graph pattern. We also distinguish c-queries without blank nodes in templates, called *blank-free*, and c-queries without GRAPH-subpatterns called *graph-free*. We use c-SPARQL to denote the class of all c-queries, and specify this notation with subscripts bf and gf for the blank- and graph-free subclasses, like c-SPARQL$_{\mathsf{bf,gf}}$.

SPARQL Semantics

The semantics of graph patterns is defined in terms of *mappings*; that is, partial functions from variables $\mathbf{V}$ to terms $\mathbf{T}$. The *domain* $\mathsf{dom}(\mu)$ of a mapping μ is the set of variables on which μ is defined. Two mappings μ_1 and μ_2 are *compatible* (written as $\mu_1 \sim \mu_2$) if $\mu_1(?x) = \mu_2(?x)$ for all variables $?x$ which are in both $\mathsf{dom}(\mu_1)$ and $\mathsf{dom}(\mu_2)$. If $\mu_1 \sim \mu_2$, then we write $\mu_1 \cup \mu_2$ for the mapping obtained by extending μ_1 according to μ_2 on all the variables in $\mathrm{dom}(\mu_2) \setminus \mathrm{dom}(\mu_1)$.

Given two sets of mappings M_1 and M_2, the *join*, *union* and *difference* between M_1 and M_2 are defined respectively as follows:

$$\begin{array}{lcl} M_1 \bowtie M_2 & = & \{\mu_1 \cup \mu_2 \mid \mu_1 \in M_1 \text{ and } \mu_2 \in M_2 \text{ such that } \mu_1 \sim \mu_2\},\\ M_1 \cup M_2 & = & \{\mu \mid \mu \in M_1 \text{ or } \mu \in M_2\},\\ M_1 \setminus M_2 & = & \{\mu_1 \mid \mu_1 \in M_1,\ \text{there is no } \mu_2 \in M_2 \text{ such that } \mu_1 \sim \mu_2\}. \end{array}$$

Based on these, the *left outer join* operation is defined as

$$M_1 \mathbin{⟕} M_2 = (M_1 \bowtie M_2) \cup (M_1 \setminus M_2).$$

Given a dataset $D = \{G_0, \langle u_1, G_1\rangle, \ldots, \langle u_n, G_n\rangle\}$, and a graph G among $G_0, \ldots, G_n$, the *evaluation* $[\![P]\!]^D_G$ of a graph pattern P over D with respect to G is defined as follows:

1. if P is a triple pattern, then $[\![P]\!]^D_G = \{\mu : \mathsf{var}(P) \to \mathbf{T} \mid \mu(P) \in G\}$,
2. if $P = (P_1\ \mathsf{AND}\ P_2)$, then $[\![P]\!]^D_G = [\![P_1]\!]^D_G \bowtie [\![P_2]\!]^D_G$,
3. if $P = (P_1\ \mathsf{OPT}\ P_2)$, then $[\![P]\!]^D_G = [\![P_1]\!]^D_G \mathbin{⟕} [\![P_2]\!]^D_G$,
4. if $P = (P_1\ \mathsf{UNION}\ P_2)$, then $[\![P]\!]^D_G = [\![P_1]\!]^D_G \cup [\![P_2]\!]^D_G$,
5. if $P = (g\ \mathsf{GRAPH}\ P')$, then either $[\![P]\!]^D_G = [\![P']\!]^D_{\mathsf{gr}_D(g)}$ in case of $g \in \mathbf{I}$, or $[\![P]\!]^D_G = \bigcup_{u\in\mathbf{I}} \left([\![P']\!]^D_{\mathsf{gr}_D(u)} \bowtie \{g \mapsto u\}\right)$ in case of $g \in \mathbf{V}$,
6. if $P = (P'\ \mathsf{FILTER}\ R)$, then $[\![P]\!]^D_G = \{\mu \mid \mu \in [\![P']\!]^D_G \text{ and } \mu \models R\}$, where a mapping μ *satisfies* a built-in condition R, denoted by $\mu \models R$, if one of the following holds:
   - R is $?x = u$, $?x \in \mathsf{dom}(\mu)$ and $\mu(?x) = u$; or
   - R is $?x = ?y$, $?x \in \mathsf{dom}(\mu)$, $?y \in \mathsf{dom}(\mu)$ and $\mu(?x) = \mu(?y)$; or
   - R is $isBlank(?x)$ and $?x \in \mathsf{dom}(\mu)$ and $\mu(?x) \in \mathbf{B}$; or
   - R is $bound(?x)$ and $?x \in \mathsf{dom}(\mu)$; or
   - R is an evaluating to true Boolean combination of other filter conditions.

The evaluation $[\![P]\!]^D$ of a pattern P over a dataset D with default graph G_0 is $[\![P]\!]^D_{G_0}$.

To define semantics of c-queries, we fix, for every template H and dataset D, a family $F(H, D)$ of *renaming functions* $f_\mu : blank(H) \to \mathbf{B} \setminus blank(D)$ parametrised by mapping μ, which are injective (i.e., there are no different b, b' such that $f_\mu(b) = f_\mu(b')$), and have pairwise disjoint ranges (i.e., there are no b, b' such that $f_{\mu_1}(b) = f_{\mu_2}(b')$ for different μ_1 and μ_2).

Finally, the *answer* $\mathsf{ans}(\boldsymbol{q}, D)$ to a c-query $\boldsymbol{q} = \mathsf{CONSTRUCT}\ H\ \mathsf{WHERE}\ P$ over an input dataset D is the RDF graph consisting of all triples in $\mu(f_\mu(H))$, for all $\mu \in [\![P]\!]^D$, which are well-formed RDF triples (i.e., have neither blank nodes on predicate positions nor variables).

3 Blank-free c-Queries

We start our study with c-SPARQL$_{\mathsf{bf}}$, that is, the language of c-queries which contain no blank nodes in their templates. This fragment is of fundamental importance, because it has simple syntax and clear intuitive semantics. All the blank nodes in the answer graph of a c-SPARQL$_{\mathsf{bf}}$ query appear in the input dataset, and, as it can be seen by careful inspection of the semantics, are treated in the same way as IRIs, with the only exception that candidate triples with blank nodes on predicate position are rejected from the answer.

The first problem we consider is the expressive power of such c-queries. As it is usually done in databases, our yardstick is first order logic (FO) with safe negation. However, since we are dealing with c-queries that input RDF graphs and datasets, it is only fair to compare them with FO over a signature that corresponds to these entities. Formally, we specify the following query language. Its input domain is the set of all finite first-order structures over elements $\mathbf{T}$ and predicates $Default$, $Named$, and $IsBlank$ of arities $3, 4$, and 1, correspondingly, such that $IsBlank(b)$ holds for some b if and only if $b \in \mathbf{B}$, and $IsBlank(b)$ implies that none of $Default(a, b, c)$, $Named(b, a, c, d)$ and $Named(d, a, b, c)$ hold for any a, c, and d. The answers for this language are sets of triples from $\mathbf{T} \times \mathbf{I} \times \mathbf{T}$, that is, essentially, RDF graphs. The set FO$_{\mathsf{rdf}}$ of queries of this language consists of all well-formed FO formulas over the signature given above. Finally, the evaluation function of FO$_{\mathsf{rdf}}$ is the usual FO entailment $\models_{\mathrm{adom}}$ over active domain semantics, that is, in particular, the quantification is realised over the finite set of all the terms from $\mathbf{T}$ appearing in the input database and query (see [1] for formal definitions).

Note that the set of input databases of FO$_{\mathsf{rdf}}$ have a straightforward one-to-one correspondence with the set of input datasets of c-SPARQL$_{\mathsf{bf}}$ queries, and the same holds for answers of these languages. Having this correspondence, we may compare their expressive power. To make such comparisons formal, we need the following definitions. A query language $\mathcal{Q}_1$ is *contained* in a language $\mathcal{Q}_2$ iff there are bijections $\mathsf{trans}_{\mathcal{I}} : \mathcal{I}_1 \to \mathcal{I}_2$, $\mathsf{trans}_{\mathcal{O}} : \mathcal{O}_1 \to \mathcal{O}_2$ between their input sets $\mathcal{I}_i$ and answer sets $\mathcal{O}_i$, and a function $\mathsf{trans}_{\mathcal{Q}} : \mathcal{Q}_1 \to \mathcal{Q}_2$ such that $\mathsf{trans}_{\mathcal{O}}(\mathsf{eval}_1(\boldsymbol{q}, I)) = \mathsf{eval}_2(\mathsf{trans}_{\mathcal{Q}}(\boldsymbol{q}), \mathsf{trans}_{\mathcal{I}}(I))$ holds for any $\boldsymbol{q} \in \mathcal{Q}_1$ and $I \in \mathcal{I}_1$, where eval_i are evaluation functions of the languages. Two languages are *equivalent* iff they contain each other.

We are ready to present our first result, claiming that the language of blank-free construct queries is subsumed by first order logic.

▶ **Lemma 3.** *The language* c-SPARQL$_{\mathsf{bf}}$ *is contained in the language* FO$_{\mathsf{rdf}}$.

This lemma can be shown using the ideas similar to the ones in the reductions from the language of SPARQL queries with result in the SELECT form, that is, the language of graph patterns described above, enriched with projection to non-recursive Datalog with safe negation, which are developed in [2] and [31]. The idea of these reductions is to assemble an extensional predicate for each sub-pattern, such that the evaluation of that predicate contains all the tuples corresponding to mappings in the evaluation of the sub-pattern. Since these mappings have different domains, the undefined value is modelled by a special constant $Null$. However, we present another reduction, where $Null$ is not used, but instead all the mappings in the result of a sub-pattern evaluation with the same domain have their

own predicate. We believe that such construction has several advantages over the first one, being simpler and intuitively more clear, so it has its own right to be given explicitly.

Proof (idea). First, we establish an equivalence between graph patterns and $\mathrm{FO}_{\mathsf{rdf}}$. Once it is done, the only remaining step is to project out the non-relevant variables and generate the desired triples. We do it as follows.

Given a graph pattern P, for every $X \subseteq \mathsf{var}(P)$ we construct a formula φ^P_X with X as free variables, such that a mapping μ is in $[\![P]\!]^D$ for a dataset D if and only if the variable assignment defined by μ satisfies $\varphi^P_{\mathsf{dom}(\mu)}$ in the $\mathrm{FO}_{\mathsf{rdf}}$ structure corresponding to D. Having such a formula for each set of variables allows us to easily perform an inductive construction. We illustrate it by the translation for patterns P of the form $(P_1 \ \mathsf{AND} \ P_2)$. Consider, for every subset X of $\mathsf{var}(P)$, the formula

$$\varphi^P_X = \bigvee_{X_1 \subseteq \mathsf{var}(P_1), X_2 \subseteq \mathsf{var}(P_2), X_1 \cup X_2 = X} \varphi^{P_1}_{X_1} \wedge \varphi^{P_2}_{X_2},$$

where $\varphi^{P_i}_{X_i}$ are the formulas constructed on the previous inductive step.

Finally, the ternary formula $\varphi_{\boldsymbol{q}}$ producing, for every dataset D, the set of triples which correspond to the answer graph to the c-query $\boldsymbol{q} = \mathsf{CONSTRUCT}\ H\ \mathsf{WHERE}\ P$ over D can be simply obtained from all φ^P_X by means of disjunction, existential quantification and checking that all the second arguments are IRIs. ◀

We illustrate this proof by means of the following example.

▶ **Example 4.** Recall the query $\boldsymbol{q}_{\mathrm{cons}}$ from Example 2. By simple inspection, we see that the domain of every mapping in the evaluation of the graph pattern is either $\{?p, ?n, ?w\}$ or $\{?p, ?n, ?w, ?e\}$. Hence, we only need to construct a formula for each of these sets, as the formulas corresponding to other subsets of $\mathrm{var}(\boldsymbol{q}_{\mathrm{cons}})$ will be contradictions. Following the construction process, we obtain

$$\begin{aligned} \varphi_{\{?p,?n,?w\}}(p,n,w) &= Def(p,\mathrm{name},n) \wedge Def(p,\mathrm{works\_at},w) \wedge \neg\exists e\ Def(p,\mathrm{mbox},e), \\ \varphi_{\{?p,?n,?w,?e\}}(p,n,w,e) &= Def(p,\mathrm{name},n) \wedge Def(p,\mathrm{works\_at},w) \wedge Def(p,\mathrm{mbox},e), \end{aligned}$$

where Def stays for $Default$ for brevity and '?' is omitted before variables to resemble the conventional FO notation. Having these, we need to create the formula $\varphi_{\boldsymbol{q}_{\mathrm{cons}}}$ that always outputs exactly the same graph as $\boldsymbol{q}_{\mathrm{cons}}$. As discussed above, this formula can be constructed by projecting out the non-relevant variables and checking that the triples are well-formed. In partiular, we obtain

$$\begin{aligned} \varphi_{\boldsymbol{q}_{\mathrm{cons}}}(x,y,z) = \ & \neg isBlank(y) \wedge \\ & \big(\ \exists p,n,w \left[\varphi_{\{?p,?n,?w\}}(p,n,w) \wedge (x = n \wedge y = \mathrm{works\_at} \wedge z = w)\right] \vee \\ & \ \ \exists p,n,w,e \left[\varphi_{\{?p,?n,?w,?e\}}(p,n,w,e) \wedge (x = n \wedge y = \mathrm{mbox} \wedge z = e)\right] \ \big). \end{aligned}$$

Our next result is the inclusion in the other direction.

▶ **Lemma 5.** *The language* $\mathrm{FO}_{\mathsf{rdf}}$ *is contained in the language* $\text{c-SPARQL}_{\mathsf{bf}}$.

Proof (idea). The proof of this lemma is an inductive construction that exploits the known idea that the difference operation on mappings can be expressed in SPARQL by means of the following application of optional matching [27]. Let

$$P_1 \ \mathsf{MINUS}\ P_2 = (P_1 \ \mathsf{OPT}\ (P_2 \ \mathsf{AND}\ (?x_1, ?x_2, ?x_3)))\ \mathsf{FILTER}\ \neg bound(?x_1),$$

where $?x_1, ?x_2$ and $?x_3$ are mentioned neither in P_1 nor in P_2. It is readily verified that $[\![P_1 \text{ MINUS } P_2]\!]_G^D = [\![P_1]\!]_G^D \setminus [\![P_2]\!]_G^D$ for any dataset D and its graph G. The construction in the proof is similar to the one in [2], where a reduction from non-recursive Datalog with safe negation to SPARQL graph patterns is provided. Note, however, that the reduction in that work is of limited applicability, because its translation function for answers is not surjective. ◀

Having these lemmas at hand, we conclude the following theorem.

▶ **Theorem 6.** *The languages* c-SPARQL$_{\mathsf{bf}}$ *and* FO$_{\mathsf{rdf}}$ *are equivalent in expressive power.*

This result (and its proof) has a couple of immediate important consequences. First, of them, obtained by straightforward inspection of the proofs of the previous lemmas, is that the language c-SPARQL$_{\mathsf{bf,gf}}$ of graph-free and blank-free c-queries which inputs are RDF datasets with only default graphs, is also equivalent to a fragment of first order logic, in particular, to the fragment of FO$_{\mathsf{rdf}}$ which does not allow for the quaternary predicate *Named* (we use FO$_{\mathsf{rdf}}^{\text{ternary}}$ to denote this fragment).

▶ **Corollary 7.** *The languages* c-SPARQL$_{\mathsf{bf,gf}}$ *and* FO$_{\mathsf{rdf}}^{\text{ternary}}$ *are equivalent in expressive power.*

Having this corollary and the preceding theorem, we may conclude, in particular, that even if the syntax of manipulating graph names in SPARQL is very different from the syntax for manipulating subjects, predicates and objects, semantically the values are treated very similarly; moreover, in terms of expressivity, everything which can be done with the first, can be done with the others, and vice versa.

The second important fact we can conclude from the established reductions is that the blank-free construct fragment of SPARQL is composable. Formally, a query language $\mathcal{Q}$ with the same input and answer sets $\mathcal{I}$, and evaluation function eval is *composable* iff for every pair $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathcal{Q}$ of queries there is another query $\boldsymbol{q} \in \mathcal{Q}$ such that $\mathsf{eval}(\boldsymbol{q}_1, \mathsf{eval}(\boldsymbol{q}_2, I)) = \mathsf{eval}(\boldsymbol{q}, I)$ for any input database $I \in \mathcal{I}$. Of course, according to this definition, there is no sense to talk about composability of the general language c-SPARQL$_{\mathsf{bf}}$ of blank-free c-queries, because it does not satisfy the condition that the sets of inputs and answers coincide: the former is datasets and the latter is graphs. Contrary, graph-free c-queries from c-SPARQL$_{\mathsf{bf,gf}}$ enjoy such a property, so we may conclude composability of c-SPARQL$_{\mathsf{bf,gf}}$ from composability of FO$_{\mathsf{rdf}}^{\text{ternary}}$.

▶ **Corollary 8.** *The language* c-SPARQL$_{\mathsf{bf,gf}}$ *is composable.*

We conclude this section with the complexity of blank-free c-queries evaluation. The lower bounds of the following result carries almost verbatim from the lower bounds in [27]. The upper bound is also very similar, the only additional initialization step is guessing the values of all the variables which are not mentioned in the template.

▶ **Proposition 9.** *The problem of checking whether a triple is in the answer to a c-query from* c-SPARQL$_{\mathsf{bf}}$ *over a dataset is* PSPACE*-complete in general and in* NLOGSPACE *if the c-query is fixed.*[4] *The bounds hold also for c-queries from* c-SPARQL$_{\mathsf{bf,gf}}$*.*

Hence, the complexity of blank-free c-queries evaluation is the same as complexity of evaluation of graph patterns, as well as of SPARQL queries with SELECT result form.

[4] The last setting is known as *data complexity* of the problem (see [37]).

4 OPT-free and Well-designed CONSTRUCT queries

The Semantic Web community has adopted the fragment of unions of well-designed graph patterns as a good practice for writing SPARQL queries. This is mainly because enforcing this property prevents users from writing graph patterns that do not agree with the open-world nature of the Semantic Web (see [27] for discussion). Furthermore, restricting to unions of well-designed graph patterns drops the (combined) complexity of evaluation from PSPACE-complete to coNP-complete, and to Σ_2^p-complete if projection is allowed. Also, several optimization techniques have been developed for the evaluation of well-designed queries (see [21,27]). In this section we study the properties of c-queries which graph patterns are unions of well-designed patterns. We concentrate on the sublanguages of c-SPARQL$_{\mathsf{bf,gf}}$, leaving c-queries with blank nodes in templates for the next section, and restricting to graph-free c-queries for brevity. Note, however, that all the relevant results in this section hold also for c-queries with GRAPH-patterns.

We start the formal part of this section with the definition of *well-designed* graph patterns, which are graph patterns with

1. no UNION-subpatterns,
2. only FILTER-subpatterns (P FILTER R) such that all variables in R are mentioned in P,
3. only OPT-subpatterns (P_1 OPT P_2) such that all variables in P_2 which appear outside this subpattern are mentioned in P_1.

If the graph pattern of a c-query is a union of well-designed patterns, then the c-query is *union-well-designed*. In this section we also consider c-queries without OPT-subpatterns, called *opt-free*. We will use superscripts uwd and of to specify sublanguages satisfying these restrictions, like c-SPARQL$^{\mathsf{uwd}}_{\mathsf{bf,gf}}$. Note, however, that even if opt-free c-queries are not union-well-designed per se, they can be easily transformed to such by applying distributivity rules to push UNION outside and techniques of [2] to enforce the condition on FILTER subpatterns. Hence, c-SPARQL$^{\mathsf{uwd}}_{\mathsf{bf,gf}}$ contains c-SPARQL$^{\mathsf{of}}_{\mathsf{bf,gf}}$. The next lemma shows that, somehow surprisingly, the containment holds in other direction as well, which means that well-designed OPT does not increase the expressive power of c-queries.

▶ **Lemma 10.** *The language* c-SPARQL$^{\mathsf{uwd}}_{\mathsf{bf,gf}}$ *is contained in the language* c-SPARQL$^{\mathsf{of}}_{\mathsf{bf,gf}}$.

This result heavily relies on the fact that those mappings from the evaluation of the graph pattern which have undefined variables from the template of c-query do not contribute to the overall answer of the c-query. This is different from the evaluation of graph patterns by themselves and SPARQL queries in SELECT result form, so such a result does not hold for those query languages.

An important corollary from the proof of the previous lemma is that the translation from a union-well-designed c-query to an opt-free c-query can be done very efficiently. This means that, under c-queries, well-designed OPT is not just dispensable, but syntactic sugar.

▶ **Corollary 11.** *Every c-query from* c-SPARQL$^{\mathsf{uwd}}_{\mathsf{bf,gf}}$ *can be transformed to an equivalent c-query from* c-SPARQL$^{\mathsf{of}}_{\mathsf{bf,gf}}$ *in* LOGSPACE.

We also relate the described languages to a fragment of first order logic, defined next. A formula $\varphi \in \mathrm{FO}^{\mathrm{ternary}}_{\mathsf{rdf}}$ is $\exists$-*positive* if it is in the $\{\exists, \neg, \vee\}$ fragment of FO and every occurrence of the predicate $Default$ is under an even number of negations. The language of all $\exists$-positive formulas is denoted $\exists\text{pos-FO}^{\mathrm{ternary}}_{\mathsf{rdf}}$.

The following result comes from a straightforward inspection of the reduction from c-SPARQL$_{\mathsf{bf}}$ to FO$_{\mathsf{rdf}}$ (Lemma 3), as the only way to generate negation over the $Default$ predicate is by means of OPT patterns.

▶ **Lemma 12.** *The language* c-SPARQL$^{\mathsf{of}}_{\mathsf{bf,gf}}$ *is contained in the language* $\exists$pos-FO$^{\text{ternary}}_{\mathsf{rdf}}$.

Quite similarly, an inspection of the proof of Lemma 5 shows that a transformation of a formula in which the predicate $Default$ appears only positively gives us a c-query which does not use the OPT operator. Note, however, that this c-query can have negations in the filter expressions.

▶ **Lemma 13.** *The language* $\exists$pos-FO$^{\text{ternary}}_{\mathsf{rdf}}$ *is contained in the language* c-SPARQL$^{\mathsf{uwd}}_{\mathsf{bf,gf}}$.

Having the previous three lemmas at hand and knowing that c-SPARQL$^{\mathsf{of}}_{\mathsf{bf,gf}}$ is contained in c-SPARQL$^{\mathsf{uwd}}_{\mathsf{bf,gf}}$, we state the main theorem of this section.

▶ **Theorem 14.** *The languages* c-SPARQL$^{\mathsf{uwd}}_{\mathsf{bf,gf}}$, c-SPARQL$^{\mathsf{of}}_{\mathsf{bf,gf}}$ *and* $\exists$pos-FO$^{\text{ternary}}_{\mathsf{rdf}}$ *are equivalent in expressive power.*

We obtain as a corollary is the composability of c-SPARQL$^{\mathsf{uwd}}_{\mathsf{bf,gf}}$.

▶ **Corollary 15.** *The language* c-SPARQL$^{\mathsf{uwd}}_{\mathsf{bf,gf}}$ *is composable.*

We conclude this section with the complexity of evaluation of union-well-designed c-queries. As with it expressive power, the complexity of evaluation for this fragment is lower than the complexity of evaluation of SELECT queries with well-designed patterns, which is, as already mentioned, Σ^p_2-complete.

▶ **Proposition 16.** *The problem of checking whether a triple is in the answer to a c-query from* c-SPARQL$^{\mathsf{uwd}}_{\mathsf{bf,gf}}$ *over a dataset is* NP*-complete.*

5 c-Queries with Blank Nodes in Templates

As mentioned, the specification of SPARQL allows for blank nodes in the definition of templates in c-queries. The idea of these blank nodes is to create fresh placeholders in the answer to a c-query which are local for every instantiation of the template.

▶ **Example 17.** Recall again the dataset from Figure 1. The c-query

```
CONSTRUCT {(_:b, manages, ?n), (?n, mbox, ?e)}
WHERE (
((?p, name, ?n) AND (?p, works_at, ?w))
         OPT   (?p, mbox, ?e)),
```

where `_:b` is a blank node, is intended to create a new blank node for each person, representing his manager. However, one must be cautious: the semantics of blank nodes in c-queries creates one blank node per each of the solutions of the query, and thus two blank nodes are created for Cristian, since there are two different solutions that assign Cristian to $?w$.

In this section we study the properties of c-queries with blank nodes in templates. Similarly to the previous section, we concentrate on c-SPARQL$_{\mathsf{gf}}$ queries, that is, the c-queries that do not use GRAPH operator and work, essentially, with RDF graphs but not datasets. However, all relevant results of this section transfer easily to the full class of c-SPARQL queries.

In order to understand the properties of c-SPARQL$_{\mathsf{gf}}$, we start with the study of its expressive power. Since queries from this class can create values from scratch, it does not make much sense to compare them with FO queries. Instead, we focus on the resemblance

of the semantics of blank nodes in template of c-SPARQL$_{\mathsf{gf}}$ queries with the one of nulls in universal solutions for data exchange problems (see [4] for a good introduction to the topic). In the following we show that this resemblance is not a coincidence, since all c-patterns from c-SPARQL$_{\mathsf{gf}}$ can be simulated by source-to-target dependencies in the context of data exchange; that is, informally, we may look at evaluating c-patterns as exchanging input graphs into the answer graphs. Furthermore, from our results on expressive power we obtain that c-SPARQL$_{\mathsf{gf}}$ captures the whole space of a certain family of source-to-target dependencies, establishing, in essence, that these two formalisms are equivalent in expressive power. One of the most important consequences of this result is non-composability of full c-queries, in contrast to the blank-free c-queries from previous sections.

To formally state and prove these results we need to recall some terminology on data-exchange. We begin by adapting the definitions of [4, 12] to our context. A *dependency* is an expression of the form

$$\forall \bar{x}\, \forall \bar{y}\, (\varphi(\bar{x}, \bar{y}) \to \exists \bar{z}\, \psi(\bar{x}, \bar{z}))\,, \tag{1}$$

where $\bar{x}, \bar{y}$, and $\bar{z}$ are disjoint tuples of variables, while φ and ψ are first-order formulas. We concentrate on the restricted class of such dependencies, called *source-to-target dependencies* (or *st-dependencies*), which are the dependencies with φ being from $\mathrm{FO}_{\mathsf{rdf}}^{\mathrm{ternary}}$ (i.e., formulas over $Default$ and $IsBlank$ relations), and ψ being a conjunction of atoms over a single ternary relation $OTriple$. In data exchange terminology, sets of st-dependencies are usually known as "mappings", but since we already use this term for solutions of graph patterns, we call them *de-mappings*, and denote DE$_{\mathsf{rdf}}$ the language of de-mappings.

The semantics of a de-mapping Σ can be defined as follows. A first-order structure over $OTriple$ and the set of terms $\mathbf{T}$ (called *target*) is a *solution under* Σ for a structure over $Default$, $IsBlank$ and $\mathbf{T}$ (called *source*), if every st-dependency from Σ holds in first-order sense for the union of the source and the target.

In data exchange one is usually interested in computing *universal solutions* for a source and a de-mapping Σ, that is, solutions with homomorphic images to all solutions. A typical way to do so is by means of the chase procedure. In traditional data exchange settings, this procedure instantiates each existential variable in $\bar{z}$ with a fresh *null* value for every application of a dependency. These nulls, essentially, have very similar semantics as blank nodes in SPARQL settings, so we define chase directly in these terms.

Formally, for our purposes, the *chase* of a source S under a de-mapping Σ is a target, constructed as follows. For every st-dependency of form (1) in Σ and every assignment $\pi : \bar{x} \cup \bar{y} \to \mathbf{T}$, such that $S \models_{\mathrm{adom}} \varphi(\pi(\bar{x}), \pi(\bar{y}))$, extend π to the variables $\bar{z}$ by $\pi(z) = b_z$ for each z from $\bar{z}$, where b_z is a fresh blank node from $\mathbf{B}$; and add the fact $OTriple(\pi(v_1), \pi(v_2), \pi(v_3))$ for each conjunct $OTriple(v_1, v_2, v_3)$ in ψ to the target, as long as $\pi(v_2)$ is not a blank node. The result of the chase is deterministic up to renaming of the introduced blank nodes, so we can consider DE$_{\mathsf{rdf}}$ as a query language with answers being the results of the chase.

We are now ready to compare the expressive power of c-queries and de-mappings.

▶ **Theorem 18.** *The languages* c-SPARQL$_{\mathsf{gf}}$ *and* DE$_{\mathsf{rdf}}$ *are equivalent in expressive power.*

It is known that de-mappings are not composable in the data exchange scenario [13]. We can adapt this argument into our context to obtain the following important negative result.

▶ **Proposition 19.** *The language* c-SPARQL$_{\mathsf{gf}}$ *is not composable.*

Having the expressive power of the general language established, next we refine the results above for its sublanguage c-SPARQL$_{\mathsf{gf}}^{\mathsf{uwd}}$ of graph-free c-queries with union-well-designed

graph patterns. Since we have shown that such queries are equivalent to positive FO, it would be reasonable to guess that c-SPARQL$^{\mathsf{uwd}}_{\mathsf{gf}}$ is equivalent to the query language given by de-mappings where every dependency (1) in it has the formula φ being a conjunction of atoms. This language is an important subclass of de-mappings, called *GLAV-mappings* (see, e.g., [12, 20]), denoted by GLAV$_{\mathsf{rdf}}$ in this paper.

Unfortunately, the following example shows that such a guess in not true.

▶ **Example 20.** Consider the c-query

```
CONSTRUCT {(_:b, p, ?x), (_:b, p, ?y)}
WHERE ((?x, p, a) OPT (?x, p, ?y)).
```

Note that here the same blank needs to be added to both of the triples in the template whenever a mapping that bounds both $?x$ and $?y$ exists. However, we also need to account for mappings that bind only $?x$. Hence, this c-query is not equivalent to the de-mapping

$$\begin{aligned} \forall x \forall y \left(Default(x,p,a) \wedge Default(x,p,y) \right. &\rightarrow \left. \exists z \left(OTriple(z,p,x) \wedge OTriple(z,p,y) \right) \right), \\ \forall x \left(Default(x,p,a) \right. &\rightarrow \left. \exists z \, OTriple(z,p,x) \right), \end{aligned}$$

because it creates additional blank nodes whenever the same pair of IRIs witnesses both dependencies. In fact, one can show that this c-query is not equivalent to any query in GLAV$_{\mathsf{rdf}}$.

In the above example both the chase of the de-mapping and the answer to the CONSTRUCT query are *homomorphically equivalent*, in the sense of [18]. It can be shown that this correspondence is not accidental, and an equivalence between GLAV$_{\mathsf{rdf}}$ and c-SPARQL$^{\mathsf{wd}}_{\mathsf{gf}}$ can be shown under this relaxed notion of equivalence between RDF graphs. Nevertheless, the following containment holds for the equivalence defined in this paper.

▶ **Proposition 21.** *The language* GLAV$_{\mathsf{rdf}}$ *is contained in* c-SPARQL$^{\mathsf{wd}}_{\mathsf{gf}}$.

Regardless, the following corollary is a consequence of the proof of Proposition 19.

▶ **Corollary 22.** *The language* c-SPARQL$^{\mathsf{wd}}_{\mathsf{gf}}$ *is not composable.*

We conclude this section with the following observation. Example 20 is problematic as we use the same blank in two triples in the CONSTRUCT template. If we disallow blanks, we can show that c-SPARQL$^{\mathsf{uwd}}_{\mathsf{bf,gf}}$ is equivalent to the setting given by *GAV-mappings*, that is, GLAV-mappings with empty tuple $\bar{z}$ of existential variables in every st-dependency (1).

6 Adding Recursion to SPARQL

Recursion is an integral part of most of the practical query languages, such as SQL99 [34]. The very recent version 1.1 of SPARQL also allows for some form of recursion, namely, *property paths* [38]. However, such recursion is of very limited form, in particular, it concentrates on (minor extensions of) two way regular path queries, a well known query language for graph databases [6]. It is possible to partially overcome this restriction by exploiting the power of *entailment regimes* like OWL 2 RL [15]. However, this formalism is also quite limited and, more important, not part of SPARQL 1.1 itself. The aim of this section is to develop syntax and semantics of a recursion operator in SPARQL and study its properties.

Before starting the formal development, we discuss what are the difficulties of introducing recursion in SPARQL and motivate it by an example. In the majority of query languages

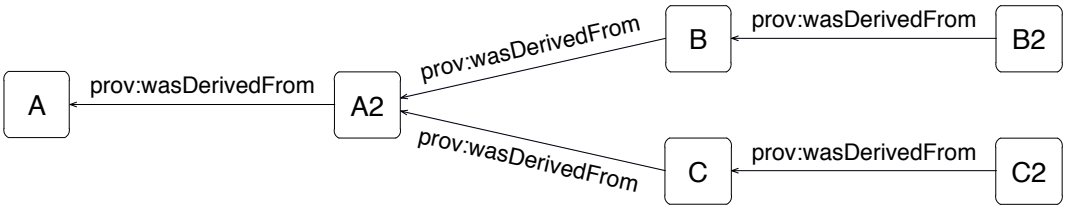


Figure 2 RDF graph storing provenance history of Wikipedia articles A, A2, B, B2, C and C2.

that allow for recursion, the semantics is defined in terms of a fixed point operator. However, to have such operator one needs to be able to pose a query over the result of another query, that is, the query language must have the same input and answer domains. Hence, it is not possible to introduce recursion based on SPARQL queries with the SELECT result form: its inputs are datasets, while its answers are sets of mappings. On the contrary, we can do it for queries with CONSTRUCT result form, since they possess this property.

In this section we show how to apply our study of c-queries in the development of a recursive operator for SPARQL. Our proposal resembles the syntax and semantics of such an operator in the SQL-99 standard. As promised, we motivate it by means of an example.

▶ **Example 23.** Consider the RDF graph in Figure 2, where a piece of the provenance information about the history of Wikipedia pages is depicted, according to PROV data model ([25]). When inspecting this graph, one of the things we may be interested in is to find all the articles that have been produced via a chain of revisions of an original article A (note that one can actually generate two different articles from a single one, and both of them would count as revisions). In SPARQL we propose to obtain all such articles by means of the following query:

```
WITH RECURSIVE http://db.ing.puc.cl/temp AS
{
  CONSTRUCT {(?x, temp:revision, A)}
  WHERE
    (?x, prov:wasDerivedFrom, A)
    UNION
    ((?x, prov:wasDerivedFrom, ?y) AND
        (http://db.ing.puc.cl/temp GRAPH (?y, temp:revision, A)))
}
SELECT ?x
WHERE (http://db.ing.puc.cl/temp GRAPH (?x, temp:revision, A)).
```

The intentional meaning of this query is as follows. The first line is the actual fixed point operator: it specifies that the RDF graph `http://db.ing.puc.cl/temp` is a temporal graph, which is iteratively computed until the least fixed point of the subsequent query is reached. In this example, the iterated query in braces essentially states that all the triples in `http://db.ing.puc.cl/temp` are of the form `(X, temp:revision, A)`, where every X is either a revision of A or linked to A via a chain of revisions of arbitrary length. Finally, the SELECT part of the query in the end extracts the desired information from the computed temporal graph `http://db.ing.puc.cl/temp`.

In this example the main query has SELECT result form, which is, in fact, not defined in this paper. In the formal exposition below, all the queries (including subqueries) have CONSTRUCT form for uniformity, but of course nothing prevents the main subquery in the end to be of any other form. We also concentrate on blank-free c-queries and leave the study of recursive c-queries with blank nodes in the templates for future work. Finally, for brevity we assume that our datasets are also blank-free, since we can handle blank nodes in exactly the same way as in the previous sections.

▶ **Definition 24.** A *recursive c-query* is either a blank-free c-query from c-SPARQL$_{\mathsf{bf}}$ or an expression of the form

$$\mathsf{WITH\ RECURSIVE}\ t\ \mathsf{AS}\ \{\boldsymbol{q}_1\}\ \boldsymbol{q}_2, \tag{2}$$

where t is an IRI from $\mathbf{I}$, $\boldsymbol{q}_1$ is a c-query from c-SPARQL$_{\mathsf{bf}}$, and $\boldsymbol{q}_2$ is a recursive c-query. The set of all recursive c-queries is denoted rec-c-SPARQL$_{\mathsf{bf}}$.

We reinforce the idea that in this definition $\boldsymbol{q}_1$ is non-recursive, but $\boldsymbol{q}_2$ could be recursive by itself, which allows us to compose recursive definitions.

Having the syntax at hand we define the semantic of recursive c-queries.

▶ **Definition 25.** Let D be a dataset, and assume that $D = D' \cup \{\langle t, G\rangle\}$, that is, that the named graph t exists in D (if t does not exists in D then let $G = \emptyset$). Then the *answer* $\mathsf{ans}(\boldsymbol{q}, D)$ of a recursive query $\boldsymbol{q}$ from rec-c-SPARQL$_{\mathsf{bf}}$ of the form (2) is equal to $\mathsf{ans}(\boldsymbol{q}_2, D_{\mathrm{lfp}})$, where D_{lfp} is the least fixed point of the sequence $D_0, D_1, \ldots$, where

$$\begin{array}{lcl} D_0 & = & D' \cup \{\langle t, G\rangle\}; \\ D_{i+1} & = & D' \cup \{\langle t, G \cup \mathsf{ans}(\boldsymbol{q}_1, D_i)\rangle\},\ \text{for } i \geq 0. \end{array}$$

Naturally, the above definition makes sense only when the sequence $D_0, D_1, \ldots$ has a (finite) fixed point. In this case, we say that the answer $\mathsf{ans}(\boldsymbol{q}, D)$ is *well-defined*. By our results on expressive power, one way to guarantee this is to require the c-query $\boldsymbol{q}_1$ to be a union of well-designed patterns, since this implies that the operator $G \cup \mathsf{ans}(\boldsymbol{q}, D_i)$ is monotone. However, we can partially relax this condition and concentrate on the following fragment of rec-c-SPARQL$_{\mathsf{bf}}$. A recursive c-query $\boldsymbol{q}$ is *semi-positive* iff it is either a simple c-query, or it is of the form (2), such that $\boldsymbol{q}_2$ is semi-positive and every subpattern P in $\boldsymbol{q}_1$ satisfies the following conditions:

1. if P is $(g\ \mathsf{GRAPH}\ P')$ with $g \in \mathbf{V} \cup \{t\}$ then P' is well-designed, and
2. if P is $(P_1\ \mathsf{OPT}\ P_2)$ then all subpatterns $(g\ \mathsf{GRAPH}\ P')$ of P_2 are such that $g \in \mathbf{I} \setminus \{t\}$.

The language of all semi-positive recursive c-queries is denoted by rec-c-SPARQL$^{\mathsf{semi}}_{\mathsf{bf}}$. They always have fixed points, as desired.

▶ **Proposition 26.** *For every recursive c-query* $\boldsymbol{q}$ *in* rec-c-SPARQL$^{\mathsf{semi}}_{\mathsf{bf}}$ *and dataset* D *the answer* $\mathsf{ans}(\boldsymbol{q}, D)$ *is well-defined.*

Having this language defined, next we study its expressive power and, in particular, show that it is equivalent to a class of Datalog programs that we call Datalog with rule-by-rule stratification (see [1] for a good introduction on Datalog).

Let V be a vocabulary of predicates, which does not contain the predicates $Default$, $Named$ (which are called *extensional* predicates) and $OTriple$ (which is called *answer predicate*). A *rule* is an expression of the form

$$Pr(\bar{x}) \leftarrow \varphi(\bar{x}, \bar{y}),$$

where $\bar{x}$ and $\bar{y}$ are tuples of variables, Pr is a predicate from $V \cup \{OTriple\}$, and φ is a conjunction of positive and negated atoms (considering equalities) over $V \cup \{Default, Named\}$ and IRIs from $\mathbf{I}$ as constants, such that every variable from $\bar{x}$ appears in φ. In such a rule, $Pr(\bar{x})$ is the *head* and $\varphi(\bar{x}, \bar{y})$ is the *body*.

A *Datalog program with rule-by-rule stratification* is a sequence $\Pi_1, \ldots, \Pi_n$ of sets of rules for which there exist a set $V = \{Pr_1, \ldots, Pr_n\}$ such that the following holds:

1. the head of each rule in Π_i is the predicate Pr_i;
2. each Π_i does not mention any Pr_j with $j > i$;
3. each Π_i does not mention Pr_i in negated atoms.

We adopt the convention that predicate Pr_n is the answer predicate of these programs, that is, $Pr_n = OTriple$. The language of all such Datalog programs is denoted $\text{Datalog}^{\mathsf{rbr}}_{\mathsf{rdf}}$. The semantics of these programs for first order structures over predicates $Default$ and $Named$ and constants $\mathbf{I}$ is the standard fixed point semantics (see, e.g., [1] for definition). We have the following result which links recursive SPARQL with this known formalism.

▶ **Theorem 27.** *The languages* $\text{rec-c-SPARQL}^{\mathsf{semi}}_{\mathsf{bf}}$ *and* $\text{Datalog}^{\mathsf{rbr}}_{\mathsf{rdf}}$ *are equivalent in expressive power.*

We conclude this section with some discussion on the relationship of the semi-positive recursive SPARQL with other known formalisms. First, from the last theorem and the results of [10] we may conclude that $\text{rec-c-SPARQL}^{\mathsf{semi}}_{\mathsf{bf}}$ contains first order logic with transitive closure. Second, it is a technicality to check that this query formalism contains SPARQL 1.1 property paths [38], c-query answering over OWL 2 RL entailment regime [15], navigational SPARQL [28], as well as GraphLog [10] and TriAL [22] query languages. Also, it is possible to show that none of these formalisms can express all $\text{rec-c-SPARQL}^{\mathsf{semi}}_{\mathsf{bf}}$ queries, that is, the containment is strict in all the cases. Hence, we may conclude that $\text{rec-c-SPARQL}^{\mathsf{semi}}_{\mathsf{bf}}$ is a clean unification of all these languages, and, as we believe, it deserves a further dedicated studies, both theoretical and applied.

7 Conclusions and Future Work

We have presented a thorough study of the expressive power and complexity of evaluation of SPARQL queries that output RDF graphs, that is, queries of CONSTRUCT form. By studying these queries we provide a strong bridge between SPARQL and well-developed frameworks like first-order logic and Datalog. In particular, we give a clean proof of the equivalence between CONSTRUCT queries and first-order logic, we characterize well-designed CONSTRUCT queries by a reduction into a positive fragment of first-order logic, and present a translation between the full fragment of CONSTRUCT queries and a specific setting for data exchange. Finally, having a good understanding of these queries we are able to present a proposal for extending SPARQL with recursion, which we prove to be equivalent in expressive power to Datalog with rule-by-rule stratification.

Queries of CONSTRUCT form are an important fragment of SPARQL provided they are the standard language to query RDF producing RDF as output. Query languages with this property have several advantages, like allowing for composability and recursion. The results in this paper present a first formal study of this fragment, and we believe the Semantic Web community will take good advantage of them. As future work we would like to extend the presented results to advance in our understanding of more expressive versions of SPARQL. There is still a good deal of research to be done in characterizing CONSTRUCT queries allowing for blank nodes in the template, as well as studying queries allowing for the GRAPH

operator. Moreover, we think our results can be extended to increase our understanding of the full-featured SPARQL, for example the expressive power of the well-designed fragment. It is also left as future work to implement the defined the recursive fragment, as well as developing and applying techniques for its optimization.

References

1. Serge Abiteboul, Richard Hull, and Victor Vianu. *Foundations of databases*, volume 8. Addison-Wesley Reading, 1995.
2. Renzo Angles and Claudio Gutierrez. The expressive power of SPARQL. In *ISWC*, pages 114–129, 2008.
3. Renzo Angles and Claudio Gutierrez. Survey of graph database models. *ACM Computing Surveys*, 40(1), 2008.
4. Marcelo Arenas, Pablo Barcelo, Leonid Libkin, and Filip Murlak. Relational and XML data exchange. *Synthesis Lectures on Data Management*, 2(1):1–112, 2010.
5. Marcelo Arenas, Sebastián Conca, and Jorge Pérez. Counting beyond a yottabyte, or how SPARQL 1.1 property paths will prevent adoption of the standard. In *Proceedings of the 21st international conference on World Wide Web*, pages 629–638. ACM, 2012.
6. Pablo Barceló Baeza. Querying graph databases. In *Proceedings of the 32nd symposium on Principles of database systems*, pages 175–188. ACM, 2013.
7. Carlos Buil-Aranda, Marcelo Arenas, and Oscar Corcho. Semantics and optimization of the SPARQL 1.1 federation extension. In *The Semanic Web: Research and Applications*, pages 1–15. Springer, 2011.
8. Melisachew Wudage Chekol, Jérôme Euzenat, Pierre Genevès, and Nabil Layaïda. SPARQL query containment under $\mathcal{SHI}$ axioms. In *AAAI*, 2012.
9. Melisachew Wudage Chekol, Jérôme Euzenat, Pierre Genevès, and Nabil Layaïda. SPARQL query containment under RDFS entailment regime. In *IJCAR*, pages 134–148, 2012.
10. Mariano P. Consens and Alberto O. Mendelzon. GraphLog: a visual formalism for real life recursion. In *Proceedings of the ninth ACM SIGACT-SIGMOD-SIGART symposium on Principles of database systems*, pages 404–416. ACM, 1990.
11. Orri Erling and Ivan Mikhailov. RDF support in the virtuoso DBMS. In *Networked Knowledge-Networked Media*, pages 7–24. Springer, 2009.
12. Ronald Fagin, Phokion G Kolaitis, Renée J Miller, and Lucian Popa. Data exchange: semantics and query answering. *Theoretical Computer Science*, 336(1):89–124, 2005.
13. Ronald Fagin, Phokion G Kolaitis, Lucian Popa, and Wang-Chiew Tan. Composing schema mappings: Second-order dependencies to the rescue. *ACM Transactions on Database Systems (TODS)*, 30(4):994–1055, 2005.
14. Floris Geerts, Grigoris Karvounarakis, Vassilis Christophides, and Irini Fundulaki. Algebraic structures for capturing the provenance of SPARQL queries. In *ICDT*, pages 153–164, 2013.
15. Birte Glimm and Chimezie Ogbuji. SPARQL 1.1 Entailment Regimes. W3C Recommendation, 2013. Available at `http://www.w3.org/TR/sparql11-entailment/`.
16. Harry Halpin and James Cheney. Dynamic provenance for SPARQL updates. In *ISWC*, 2014.
17. Steve Harris, Nick Lamb, and Nigel Shadbolt. 4store: The design and implementation of a clustered rdf store. In *5th International Workshop on Scalable Semantic Web Knowledge Base Systems (SSWS2009)*, pages 94–109, 2009.
18. Aidan Hogan, Marcelo Arenas, Alejandro Mallea, and Axel Polleres. Everything you always wanted to know about blank nodes. *Web Semantics: Science, Services and Agents on the World Wide Web*, 2014.
19. Egor V. Kostylev and Bernardo Cuenca Grau. On the semantics of SPARQL queries with optional matching under entailment regimes. In *ISWC*, 2014.
20. Maurizio Lenzerini. Data integration: A theoretical perspective. In *Proceedings of the twenty-first ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 233–246. ACM, 2002.

21 Andrés Letelier, Jorge Pérez, Reinhard Pichler, and Sebastian Skritek. Static analysis and optimization of semantic web queries. *ACM Trans. Database Syst.*, 38(4):25, 2013.

22 Leonid Libkin, Juan Reutter, and Domagoj Vrgoč. TriAL for RDF: adapting graph query languages for RDF data. In *Proceedings of the 32nd symposium on Principles of database systems*, pages 201–212. ACM, 2013.

23 Katja Losemann and Wim Martens. The complexity of evaluating path expressions in SPARQL. In *Proceedings of the 31st symposium on Principles of Database Systems*, pages 101–112. ACM, 2012.

24 Frank Manola and Eric Miller. RDF Primer. W3C Recommendation, 10 February 2004. Available at `http://www.w3.org/TR/2004/REC-rdf-primer-20040210/`.

25 Paolo Missier, Khalid Belhajjame, and James Cheney. The w3c prov family of specifications for modelling provenance metadata. In *Proceedings of the 16th International Conference on Extending Database Technology*, pages 773–776. ACM, 2013.

26 Jorge Pérez, Marcelo Arenas, and Claudio Gutierrez. Semantics and complexity of SPARQL. In *ISWC*, pages 30–43, 2006.

27 Jorge Pérez, Marcelo Arenas, and Claudio Gutierrez. Semantics and complexity of SPARQL. *ACM Trans. Database Syst.*, 34(3), 2009.

28 Jorge Pérez, Marcelo Arenas, and Claudio Gutierrez. nSPARQL: A navigational language for RDF. *Web Semantics: Science, Services and Agents on the World Wide Web*, 8(4):255–270, 2010.

29 François Picalausa and Stijn Vansummeren. What are real SPARQL queries like? In *SWIM*, 2011.

30 Reinhard Pichler and Sebastian Skritek. Containment and equivalence of well-designed SPARQL. In *Proceedings of the 33rd ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 39–50. ACM, 2014.

31 Axel Polleres and Johannes Peter Wallner. On the relation between SPARQL1.1 and answer set programming. *Journal of Applied Non-Classical Logics*, 23(1-2):159–212, 2013.

32 Eric Prud'hommeaux and Andy Seaborne. SPARQL query language for RDF. W3C Recommendation, 2008. Available at `http://www.w3.org/TR/rdf-sparql-query/`.

33 Eric Prud'hommeaux, Andy Seaborne, et al. SPARQL query language for RDF. 2006.

34 Raghu Ramakrishnan, Johannes Gehrke, and Johannes Gehrke. *Database management systems*, volume 3. McGraw-Hill New York, 2003.

35 Michael Schmidt, Michael Meier, and Georg Lausen. Foundations of SPARQL query optimization. In *ICDT*, pages 4–33, 2010.

36 Andy Seaborne. ARQ-A SPARQL processor for Jena. *Obtained through the Internet: http://jena. sourceforge. net/ARQ/*, 2010.

37 Moshe Y Vardi. The complexity of relational query languages. In *Proceedings of the fourteenth annual ACM symposium on Theory of computing*, pages 137–146. ACM, 1982.

38 W3C SPARQL Working Group. SPARQL 1.1 Query language. W3C Recommendation, 21 March 2013. Available at `http://www.w3.org/TR/sparql11-query/`.

APPENDIX: Proofs and Intermediate Results

Proof of Lemma 3

The definition of $\mathsf{trans}_{\mathcal{I}}$ and $\mathsf{trans}_{\mathcal{O}}$ are straightforward, so we concentrate on the translation $\mathsf{trans}_{\mathcal{Q}}$ for queries.

We assume an arbitrary order $\leq$ in $\mathbf{V}$. Given $X \subseteq \mathbf{V}$, we denote by $\bar{X}$ the tuple containing the variables in X ordered under $\leq$. Furthermore, for every mapping μ, if $X \subseteq \mathrm{dom}(\mu)$ we denote by $\mu(\bar{X})$ the tuple that results from replacing every component of $\bar{X}$ by its image under μ. We abuse notation by indistinctly using FO and SPARQL variables.

▶ **Lemma 28.** *For every graph pattern P there is a set of formulae $\{\varphi_X(\bar{X})\}_{X \subseteq var(P)}^{P}$ such that, for every mapping μ and dataset D, it is the case that $\mu \in [\![P]\!]^D$ if and only if $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_X^P(\mu(\bar{X}))$ with $X = dom(\mu)$.*

Proof. Let P be a SPARQL pattern. We proceed by induction on the structure of P.

- Let P be a triple pattern. Since for every dataset D and mapping $\mu \in [\![P]\!]^D$ we have $\mathrm{var}(P) = \mathrm{dom}(\mu)$, define $\varphi_X^P(\bar{X})$ as a contradiction for every $X \subsetneq \mathrm{var}(P)$. For $X = \mathrm{var}(P)$ define $\varphi_X^P(\bar{X})$ as $Default(P)$, where P is considered as a first order tuple. As usual, we are assuming that every IRI can be referred to as a constant. It readily follows that a mapping μ belongs to $[\![P]\!]_G$ if and only if $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_{\mathrm{dom}(\mu)}^P(\mu(\bar{X}))$.
- Let $P = P_1$ UNION P_2. For every $X \subseteq \mathrm{var}(P)$ define the formula $\varphi_X^P(\bar{X})$ as
$$\varphi_X^P(\bar{X}) = \varphi_X^{P_1}(\bar{X}) \vee \varphi_X^{P_2}(\bar{X}).$$
Let μ be a mapping and $X = \mathrm{dom}(\mu)$. By semantics, $\mu \in [\![P]\!]^D$ if and only if $\mu \in [\![P_1]\!]^D \cup [\![P_2]\!]^D$. Hence, by hypothesis we have $\mu \in [\![P]\!]^D$ if and only if $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_X^{P_1}(\mu(\bar{X}))$ or $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_X^{P_2}(\mu(\bar{X}))$, which is the semantic definition of $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_X^{P_1}(\mu(\bar{X})) \vee \varphi_X^{P_2}(\mu(\bar{X}))$.
- Let $P = P_1$ AND P_2. For every $X \subseteq \mathrm{var}(P)$ define the formula $\varphi_X^P(\bar{X})$ as
$$\varphi_X^P(\bar{X}) = \bigvee_{X_1 \cup X_2 = X} \left[\varphi_{X_1}^{P_1}(\bar{X}_1) \wedge \varphi_{X_2}^{P_2}(\bar{X}_2)\right].$$
Let D and μ be a dataset and a mapping, respectively, and let $X = \mathrm{dom}(\mu)$. If μ belongs to $[\![P]\!]^D$, then there are two compatible mappings $\mu_1 \in [\![P_1]\!]^D$ and $\mu_2 \in [\![P_2]\!]^D$ such that $\mu = \mu_1 \cup \mu_2$. Let $X_1 = \mathrm{dom}(\mu_1)$ and $X_2 = \mathrm{dom}(\mu_2)$. By hypothesis, we know that $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_{X_1}^{P_1}(\mu_1(\bar{X}_1))$ and $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_{X_2}^{P_2}(\mu_2(\bar{X}_2))$ which is equivalent to $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_{X_1}^{P_1}(\mu_1(\bar{X}_1)) \wedge \varphi_{X_2}^{P_2}(\mu_2(\bar{X}_2))$. As $X_1 \cup X_2 = X$ we have $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_X^P(\mu(\bar{X}))$.
For the converse, if $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_{\mathrm{dom}(\mu)}^P(\mu(\bar{X}))$ then there are two sets X_1 and X_2 such that $X_1 \cup X_2 = \mathrm{dom}(\mu)$ and both $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_{X_1}^{P_1}(\mu(\bar{X}_1))$ and $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_{X_2}^{P_1}(\mu(\bar{X}_2))$ hold. Define μ_i as μ restricted to X_i ($i \in \{1, 2\}$). It follows from the hypothesis that $\mu_1 \in [\![P_1]\!]^D$ and $\mu_2 \in [\![P_2]\!]^D$. Since μ_1 and μ_2 are compatible and $\mu = \mu_1 \cup \mu_2$, this implies $\mu \in [\![P]\!]^D$.
- Let $P = P_1$ OPT P_2. For every $X \subseteq \mathrm{var}(P)$ define the formula $\varphi_X^P(\bar{X})$ as
$$\varphi_X^P(\bar{X}) = \varphi_X^{P_1 \mathsf{AND} P_2}(\bar{X}) \vee \varphi_{\mathsf{MINUS},X}^P(\bar{X})$$
Where
$$\varphi_{\mathsf{MINUS},X}^P(\bar{X}) = \left[\varphi_X^{P_1}(\bar{X}) \wedge \neg \bigvee_{X' \subseteq \mathrm{var}(P_2)} \exists (X' \setminus X) \varphi_{X'}^{P_2}(\bar{X}')\right].$$

Let D and μ be a dataset and a mapping, respectively, and let $X = \text{dom}(\mu)$. By definition, μ belongs to $[\![P_1 \text{ AND } P_2]\!]^D$ or to $[\![P_1]\!]^D \setminus [\![P_2]\!]^D$. In the first case, we know that $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_X^{P_1 \text{AND} P_2}(\mu(\bar{X}))$. Next we show that if $\mu \in [\![P_1]\!]^D \setminus [\![P_2]\!]^D$ then $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi^P_{\mathsf{MINUS},X}(\mu(\bar{X}))$. As $\mu \in [\![P_1]\!]^D$, we know $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_X^{P_1}(\mu(\bar{X}))$, so we only need to prove that there is no set $X' \subseteq \text{var}(P_2)$ such that $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_{X'}^{P_2}(\mu'(\bar{X'}))$ for some μ' compatible with μ. But if this was the case then μ' would be in $[\![P_2]\!]^D$, which contradicts the fact that $\mu \in [\![P_1]\!]^D \setminus [\![P_2]\!]^D$ (since μ' and μ are compatible).
For the converse, assume $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_X^P(\mu(\bar{X}))$ where $X = \text{dom}(\mu)$. If $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_X^{P_1 \text{AND} P_2}(\mu(\bar{X}))$, we know by the AND case that $\mu \in [\![P_1 \text{ AND } P_2]\!]^D$ and hence $\mu \in [\![P]\!]^D$. The remaining case is when $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi^P_{\mathsf{MINUS},X}(\mu(\bar{X}))$. If this is the case, by hypothesis it readily follows that $\mu \in [\![P_1]\!]^D$. Now we have to prove that μ is not compatible with any mapping in $[\![P_2]\!]^D$. Proceed now by contrapositive. Assume there is a mapping $\mu' \in [\![P_2]\!]^D$ compatible with μ. We know $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_{X'}^{P_2}(\mu'(\bar{X'}))$ where $X' = \text{dom}(\mu')$. Since μ and μ' are compatible, the assignments in μ' can be obtained by extending those in μ, and thus $\mathsf{trans}_{\mathcal{I}}(D)$ would not satisfy $\varphi^P_{\mathsf{MINUS},X}(\mu(\bar{X}))$.

- Let $P = P_1 \text{ FILTER } R$. For every $X \subseteq \text{var}(P)$ define $\varphi_X^P(\bar{X})$ as

$$\varphi_X^P(\bar{X}) = \varphi_X^{P_1}(\bar{X}) \wedge \varphi_R(\bar{X})$$

where $\varphi_R(\bar{X})$ is inductively defined as follows:
  - If R is an equality and $\text{var}(R) \not\subseteq X$, then $\varphi_R = False$.
  - If R is an equality and $\text{var}(R) \subseteq X$, then $\varphi_R = R$.
  - If $R = isBlank(x)$ then $\varphi_R = Blank(x)$.
  - If $R = bound(x)$ and $x \notin X$ then $\varphi_R = False$.
  - If $R = bound(x)$ and $x \in X$ then $\varphi_R = True$.
  - If R is of the form $\neg R_1$, $R_1 \wedge R_2$, or $R_1 \vee R_2$ for filter conditions R_1 and R_2, then φ_R is the corresponding boolean combination of φ_{R_1} and φ_{R_2}.

  Let D and μ be a dataset and a mapping, respectively, and let $X = \text{dom}(\mu)$. It is easy to see from the definition of φ_R that $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_R(\mu(\bar{X}))$ if and only if $\mu \models R$. By hypothesis we have $\mu \in [\![P_1]\!]^D$ if and only if $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_X^{P_1}(\mu(\bar{X}))$, and hence it readily follows that $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_X^{P_1}(\mu(\bar{X})) \wedge \varphi_R(\bar{X})$ if and only if $\mu \in [\![P_1]\!]^D$ and $\mu \models R$.
- Let $P = g \text{ GRAPH } P_1$. We distinguish two cases: $g \in \mathbf{V}$ or $g \in \mathbf{I}$. If $g \in \mathbf{I}$, for every $X \subseteq \text{var}(P)$ define $\varphi_X^P(\bar{X})$ as the result of replacing in $\varphi_X^{P_1}(\bar{X})$ every occurence of $Default(t_1, t_2, t_3)$ by $Named(g, t_1, t_2, t_3)$. In the other case, if $g \in \mathbf{V}$, we know that g will be in the domain of every mapping in $[\![P]\!]^D$. Hence, for every $X \subseteq \text{var}(P)$ define $\varphi^P_{X \cup \{g\}}$ as the result of replacing in $\varphi_X^{P_1}(\bar{X})$ every occurence of $Default(t_1, t_2, t_3)$ by $Named(g, t_1, t_2, t_3)$. The result readily follows from the induction hypothesis and the semantic definition of the GRAPH operator.

◀

Now we are ready to prove our main result. Let

$$Q = \text{CONSTRUCT } \{t_1, \ldots, t_n\} \text{ WHERE } P,$$

and let $\{\varphi_X^P(\bar{X})\}_{X \subseteq \text{var}(P)}$ be the formulas obtained by the previous lemma. Let x, y, z be three variables not mentioned in any φ_X^P. We construct a formula $\varphi_Q(x, y, z)$ which is the exact translation of Q, this is, for every dataset D, $\mathsf{ans}(Q, D)$ is the set of triples (a, b, c) such that $\mathsf{trans}_{\mathcal{I}}(D) \models \varphi_Q(a, b, c)$. We first do this separately for every t_i, and then gather

them by means of disjunction. Let $t \in \{t_1, \ldots, t_n\}$. We denote the i-th entry of t by $t[i]$. For every $X \subseteq \text{var}(P)$ define the formula $\psi^P_{X,t}$ as

$$\psi^P_{X,t}(x,y,z) = \begin{cases} \exists X(\varphi^P_X \wedge x = t[1] \wedge y = t[2] \wedge z = t[3]) & \text{if } \text{var}(t) \subseteq X \\ x \neq x & \text{otherwise.} \end{cases}$$

It is easy to see that this formula outputs every triple generated by t and a mapping which domain is X, including those containing blank nodes as properties. Finally, we construct the promised formula as

$$\varphi_q(x,y,z) = \neg isBlank(y) \wedge \bigvee_{t \in \{t_1 \ldots, t_n\}} \bigvee_{X \subseteq \text{var}(P)} \psi^P_{X,t}.$$

This formula outputs every well-formed triple in Q generated by some t and some mapping in the answer to P, which concludes our proof.

Proof of Lemma 5

The definition of $\mathsf{trans}_{\mathcal{I}}$ and $\mathsf{trans}_{\mathcal{O}}$ are straightforward, so we concentrate on the translation $\mathsf{trans}_{\mathcal{Q}}$ for queries. We first inductively define a translation from formulas φ, which satisfy all the conditions of FO_{rdf}, except, maybe, that it can be of arbitrary arity, to graph patterns P_φ. Since $IsBlank(b)$ holds for and only for b in $\mathbf{B}$, we may assume that φ does not contain atoms of the form $IsBlank(u)$ for $u \in \mathbf{T}$. Having

$$\begin{aligned} ADom^1(?x) \quad = \quad & ((?x, ?x^2_1, ?x^3_1) \text{ UNION } (?x^1_2, ?x, ?x^3_2) \text{ UNION } (?x^1_3, ?x^2_3, ?x) \text{ UNION} \\ & (?x \text{ GRAPH } (?x^2_4, ?x^3_4, ?x^4_4)) \text{ UNION } (?x^1_5 \text{ GRAPH } (?x, ?x^3_5, ?x^4_5)) \text{ UNION} \\ & (?x^1_6 \text{ GRAPH } (?x^2_6, ?x, ?x^4_6)) \text{ UNION } (?x^1_7 \text{ GRAPH } (?x^2_7, ?x^3_7, ?x^4_7))) \end{aligned}$$

for fresh variables $?x^j_i$, and $ADom(?x_1, \ldots, ?x_n) = (?y_1, ?y_2, ?y_3)$ AND $ADom^1(?x_1)$ AND $\ldots$ AND $ADom^1(?x_n)$ for fresh $?y_i$, we define

1. if φ is $Default(t_1, t_2, t_3)$ then $P_\varphi = (t_1, t_2, t_3)$;
2. if φ is $Named(t, t_1, t_2, t_3)$ then $P_\varphi = (t \text{ GRAPH } (t_1, t_2, t_3))$;
3. if φ is $IsBlank(?x)$ then $P_\varphi = (ADom(?x) \text{ FILTER } isBlank(?x))$;
4. if φ is $?x = ?y$ then $P_\varphi = (ADom(?x, ?y) \text{ FILTER } ?x = ?y)$;
5. if φ is $?x = u$ then $P_\varphi = (ADom(?x) \text{ FILTER } ?x = u)$;
6. if φ is $\varphi_1 \wedge \varphi_2$ then $P_\varphi = (P_{\varphi_1} \text{ AND } P_{\varphi_2})$;
7. if φ is $\varphi_1 \vee \varphi_2$ then $P_\varphi = (P_{\varphi_1} \text{ UNION } P_{\varphi_2})$;
8. if φ is $\neg\varphi'$ then $P_\varphi = (ADom(\mathsf{free}(\varphi')) \text{ MINUS } P_{\varphi'})$;
9. if φ is $\exists ?x\ \varphi'$ then $P_\varphi = P_{\varphi'}$,

where $\mathsf{free}(\varphi)$ is the set of free variables of φ. Then we define $\mathsf{trans}_{\mathcal{Q}}(\varphi)$ for an FO_{rdf} formula $\varphi(?x, ?y, ?z)$ as CONSTRUCT $\{(?x, ?y, ?z)\}$ WHERE $(P_\varphi \text{ FILTER } \neg isBlank(?y))$.

Proof of Proposition 9

The PSPACE lower bound follows directly from the PSPACE lower bound for graph patterns using only AND, FILTER and OPT presented in [27]. In that reduction, given a quantified boolean formula φ they construct a SPARQL graph pattern P_φ, an RDF graph G_φ and a mapping μ such that φ is valid if and only if $\mu \in [\![P_\varphi]\!]_{G_\varphi}$. The key facts are that the mapping μ is independent of φ and, moreover, if $\mu \in [\![P_\varphi]\!]_{G_\varphi}$ then μ is the only mapping in $[\![P_\varphi]\!]_{G_\varphi}$ assigning $\mu(?x)$ to $?x$. Hence, given a quantified boolean formula φ, we create the c-query

$$Q_\varphi = \text{CONSTRUCT } \{(?x, ?x, ?x)\} \text{ WHERE } P_\varphi$$

where $?x \in \mathrm{dom}(\mu)$. Then it readily follows that φ is valid if and only if the triple $(\mu(?x), \mu(?x), \mu(?x))$ belongs to $\mathsf{ans}(Q_\varphi, \langle G_\varphi \rangle)$.

For the PSPACE upper bound, we extend the well-known fact that evaluation of SPARQL graph patterns is in PSPACE. Let T be a triple, $Q = \mathsf{CONSTRUCT}\ H\ \mathsf{WHERE}\ P$ a c-query in c-SPARQL$_{\mathsf{bf}}$ and D a dataset. To know if T belongs to $\mathsf{ans}(Q, D)$, we simply choose a triple $t \in H$ and a mapping μ such that $\mu(t) = T$. Then, we verify that $\mu \in [\![P]\!]^D$. All of the previous steps can be done in NPSPACE and hence in PSPACE [27].

For the NLOGSPACE upper bound let $Q = \mathsf{CONSTRUCT}\ H\ \mathsf{WHERE}\ P$ be a fixed c-query. Given an RDF cataset D and a triple T, to know if $T \in \mathsf{ans}(Q, D)$ we choose a triple $t \in H$ and a mapping μ such that $\mu(t) = T$ and then we check if $\mu \in [\![P]\!]^D$. Since the query Q is fixed, all the previous steps can be done in NLOGSPACE [27].

Proof of Lemma 10

We consider every pattern in this proof to be GRAPH-free. We need to prove that for every c-query Q in c-SPARQL$^{\mathsf{uwd}}_{\mathsf{bf,gf}}$ there is a c-query in c-SPARQL$^{\mathsf{of}}_{\mathsf{bf,gf}}$ which is equivalent to Q. To do so, we recall some definitions from [27]. Given two graph patterns P and P', P' is said to be a direct reduction of P if P' can be obtained from P by replacing a subformula $P_1\ \mathsf{OPT}\ P_2$ by P_1. The reflexive and transitive closure of this relation is denoted by $\trianglelefteq$. For every pattern P, and(P) is the result of replacing in P every OPT by AND. Given a pattern P and an RDF graph G, a mapping μ is said to be a partial solution to P over G if there is a graph pattern P' such that $P' \trianglelefteq P$ and $\mu \in [\![\mathrm{and}(P')]\!]_G$.

We need to introduce some further notation. For every pattern P, define P_{of} as the result of replacing in P every subpattern $P_1\ \mathsf{OPT}\ P_2$ by $P_1\ \mathsf{UNION}\ (P_1\ \mathsf{AND}\ P_2)$.

▶ **Proposition 29.** *For every RDF graph G and graph pattern P, $[\![P_{\mathsf{of}}]\!]_G$ is the set of partial solutions to P over G.*

Proof. We start by showing that every partial solution to P over G is in $[\![P_{\mathsf{of}}]\!]_G$. Let μ be a partial solution to P over G. We proceed by induction over P.

- If $P = P_1\ \mathsf{UNION}\ P_2$, then without loss of generality we can assume that μ is a partial solution to P_1 over G. Then μ belongs to $[\![P_{1\mathsf{of}}]\!]_G$, and hence to $[\![P_{1\mathsf{of}}\ \mathsf{UNION}\ P_{2\mathsf{of}}]\!]_G = [\![P_{\mathsf{of}}]\!]_G$.
- If $P = P_1\ \mathsf{AND}\ P_2$, then there are two mappings μ_1 and μ_2, with $\mu = \mu_1 \cup \mu_2$, which are partial solutions to P_1 and to P_2 over G, respectively. By hypothesis, $\mu_1 \in [\![P_{1\mathsf{of}}]\!]_G$ and $\mu_2 \in [\![P_{2\mathsf{of}}]\!]_G$. Hence, $\mu \in [\![P_{1\mathsf{of}}\ \mathsf{AND}\ P_{2\mathsf{of}}]\!]_G = [\![P_{\mathsf{of}}]\!]_G$.
- If $P = P_1\,\mathsf{FILTER}\,R$, then μ is a partial solution to P_1 which satisfies R. Hence, μ belongs to $[\![P_{1\mathsf{of}}\ \mathsf{FILTER}\ R]\!]_G = [\![P_{\mathsf{of}}]\!]_G$.
- If $P = P_1\ \mathsf{OPT}\ P_2$, either μ is a partial solution to P_1 over G or μ is a partial solution to $P_1\ \mathsf{AND}\ P_2$ over G. Then, we know by hypothesis that $\mu \in [\![P_{1\mathsf{of}}]\!]_G$ or $\mu \in [\![P_{1\mathsf{of}}\ \mathsf{AND}\ P_{2\mathsf{of}}]\!]_G$. In either case we have $\mu \in [\![P_{1\mathsf{of}}\ \mathsf{UNION}\ (P_{1\mathsf{of}}\ \mathsf{AND}\ P_{2\mathsf{of}})]\!]_G = [\![P_{\mathsf{of}}]\!]_G$.

Next we prove that every mapping $\mu \in [\![P_{\mathsf{of}}]\!]_G$ is a partial solution to P over G. Again we proceed by induction on P. If P is a triple, UNION-, AND- or FILTER-pattern, the result readily follows by the induction hypothesis as in the previous case. If $P = P_1\ \mathsf{OPT}\ P_2$, then we know that $P_{\mathsf{of}} = P_{1\mathsf{of}}\ \mathsf{UNION}\ (P_{1\mathsf{of}}\ \mathsf{AND}\ P_{2\mathsf{of}})$. Therefore, μ may either be in $[\![P_{1\mathsf{of}}]\!]_G$ or in $[\![P_{1\mathsf{of}}\ \mathsf{AND}\ P_{2\mathsf{of}}]\!]_G$. Hence, we know μ is a partial result to P_1 over G or to $P_1\ \mathsf{AND}\ P_2$ over G. In both cases, by definition it is a partial solution to $P_1\ \mathsf{OPT}\ P_2$ over G.

◀

We recall a proposition from [27], for which we need the next definition. Given two mappings μ and μ', μ' is said to be subsumed by μ if both $\mathrm{dom}(\mu') \subseteq \mathrm{dom}(\mu)$ and $\mu_1 \sim \mu_2$ hold. Given a set of mappings M, a mapping $\mu \in M$ is said to be maximal (in M) if there is no $\mu' \in M \setminus \{\mu\}$ subsuming μ.

▶ **Proposition 30** (Pérez, Arenas, Gutiérrez [27]). *Given a UNION-free well-designed pattern P and an RDF graph G, for every mapping μ it is the case that $\mu \in [\![P]\!]_G$ if and only if μ is a maximal partial solution for P over G.*

▶ **Lemma 31.** *For every well-designed pattern P and every RDF graph G, the maximal mappings in $[\![P]\!]_G$ are the same as the maximal mappings in $[\![P_{\mathsf{of}}]\!]_G$.*

Proof. Let μ be a maximal mapping in $[\![P]\!]_G$. In particular it is a maximal mapping in $[\![P']\!]_G$ for some UNION-free disjunct P' of P. By Proposition 30, μ is a partial solution to P' over G, and hence it is a partial solution to P over G. We conclude by Proposition 29 that $\mu \in [\![P_{\mathsf{of}}]\!]_G$.

Let μ be a maximal mapping in $[\![P_{\mathsf{of}}]\!]_G$. Since $[\![P_{\mathsf{of}}]\!]_G$ is the set of partial solutions to P over G (Proposition 29), we know μ is a maximal partial solution to P over G. Hence, it must be a maximal partial solution to P' over G for some disjunct P' of P. By Proposition 30 we conclude that $\mu \in [\![P']\!]_G$ and hence $\mu \in [\![P]\!]_G$. ◀

Now we have all the necessary ingredients to prove the main result. Let $Q = \mathsf{CONSTRUCT}\ H\ \mathsf{WHERE}\ P$ be a c-query in c-SPARQL$^{\mathsf{uwd}}_{\mathsf{bf,gf}}$. Define Q_{of} as $\mathsf{CONSTRUCT}\ H\ \mathsf{WHERE}\ P_{\mathsf{of}}$. We prove that for every rdf graph G it is the case that $\mathsf{ans}(Q, G) = \mathsf{ans}(Q_{\mathsf{of}}, G)$.

Let T be a triple in $\mathsf{ans}(Q_{\mathsf{of}}, G)$. We know there is a mapping $\mu \in [\![P_{\mathsf{of}}]\!]_G$ and a triple $t \in H$ such that (1) $\mathrm{var}(t) \subseteq \mathrm{dom}(\mu)$, (2) $\mu(t) = T$, and (3) $\mu(t)$ does not contain a blank node as second component. It is easy to see that every mapping μ' subsuming μ satisfies the previous three properties. But by Lemma 31 there must be a mapping μ' subsuming μ in $\mu \in [\![P]\!]_G$, and hence $T = \mu'(t) \in \mathsf{ans}(Q, G)$. The other direction is obtained by the exact same analysis but changing $\mathsf{ans}(Q, G)$ by $\mathsf{ans}(Q_{\mathsf{of}}, G)$.

Proof of Proposition 16

Since we can translate between c-SPARQL$^{\mathsf{uwd}}_{\mathsf{bf,gf}}$ and c-SPARQL$^{\mathsf{of}}_{\mathsf{bf,gf}}$ in LOGSPACE, we prove the property for c-queries in c-SPARQL$^{\mathsf{of}}_{\mathsf{bf,gf}}$. We rely on the well-known fact that the problem of, given an RDF dataset D, a mapping μ and an OPT-free graph pattern P, determiing if $\mu \in [\![P]\!]^D$, is NP-complete [26]. Let T be a triple, $Q = \mathsf{CONSTRUCT}\ H\ \mathsf{WHERE}\ P$ a c-query in c-SPARQL$^{\mathsf{of}}_{\mathsf{bf,gf}}$ and D a dataset. To know if T belongs to $\mathsf{ans}(Q, D)$, we simply choose a triple $t \in H$ and a mapping μ such that $\mu(t) = T$. Then, we verify that $\mu \in [\![P]\!]^D$. By the previously mentioned result, all of the previous steps can be done in NP.

For the lower bound, we inspect the reduction used in [26]. In that reduction, given a propositional formula φ they construct an OPT-free SPARQL graph pattern P_φ, an RDF graph G_φ and a mapping μ_φ such that φ is satisfiable if and only if $\mu_\varphi \in [\![P_\varphi]\!]_{G_\varphi}$. The key fact is that if $\mu_\varphi \in [\![P]\!]_{G_\varphi}$ then $[\![P]\!]_{G_\varphi} = \{\mu_\varphi\}$, and if $\mu_\varphi \notin [\![P]\!]_{G_\varphi}$ then $[\![P]\!]_{G_\varphi} = \emptyset$. Hence, given a propositional formula φ, we create the c-query

$$Q_\varphi = \mathsf{CONSTRUCT}\ \{(?x, ?x, ?x)\}\ \mathsf{WHERE}\ P_\varphi$$

where $?x \in \mathrm{dom}(\mu_\varphi)$. Then it readily follows that φ is satisfiable if and only if the triple $(\mu_\varphi(?x), \mu_\varphi(?x), \mu_\varphi(?x))$ belongs to $\mathsf{ans}(Q_\varphi, \langle G_\varphi \rangle)$.

Proof of Proposition 18

Assume we have a query of form $C = \mathsf{CONSTRUCT}\ \{t_1, \ldots, t_n\}\mathsf{WHERE}P$. For each $V = \{x_1, \ldots, x_n\} \subseteq \text{var}(P)$, let $\text{var}(P) \setminus V = \{y_1, \ldots, y_m\}$, and T_V all triples mentioning only blanks and variables from V. Furthermore, let C_V be the following pattern:

$$C_V = \mathsf{CONSTRUCT}\ T_V\ \mathsf{WHERE}\ P\ \mathsf{FILTER}$$
$$?x_1 = ?x_1\ \mathsf{AND} \cdots ?x_n = ?x_n\ \mathsf{AND}\ NotBound(y_1)\ \mathsf{AND} \cdots \mathsf{AND}\ NotBound(y_m) \quad (3)$$

We now claim

▶ **Lemma 32.** *For all RDF graph G, the graph $\mathsf{ans}(C, G)$ is equivalent to the union of $\{\mathsf{ans}(C_V, G) \mid V \subseteq var(P)\}$, up to renaming of blanks*

Proof. If a triple (a, b, c) belongs to $\mathsf{ans}(C, G)$, then there is a mapping $\mu \in [\![P]\!]_G$ and a triple $t \in \{t_1, \ldots, t_n\}$ such that $\text{var}(t) \subseteq \text{dom}(\mu)$ and $\mu(t) = (a, b, c)$. Let $V = \text{dom}(\mu)$. Notice that t must belong to T_V, and μ satisfies the filter condition of C_V. Hence, (a, b, c) also belongs to $\mathsf{ans}(C_V, G)$.

The other direction is analogous: every mapping μ witnesses at most one such C_V, and all mappings witnessing any of the C_V's must also witness C. ◀

For each such C_V, construct FO formula $\varphi_V^P(\bar{V})$ as shown in Lemma 3. Furthermore, let $f_V : \mathbf{B} \cup \mathbf{V} \to \mathbf{V}$ be a function that is the identity on $\mathbf{V}$ and that replaces each blank in T_V for a fresh variable in $\mathbf{V}$.

Let ϕ_{T_V} be the conjunction of $Default(f(a), f(b), f(c))$ for each triple (a, b, c) in T_V.

Then Σ contains, for each $V = \{x_1, \ldots, x_n\} \subseteq \text{var}(P)$, the dependency

$$\forall x_1 \cdots \forall x_n \big(\varphi_V^P(\bar{X}) \to \exists \bar{z} \phi_{T_V}\big),$$

where $\bar{z}$ is the tuple of all variables created by f_V.

From Lemma 32 and Lemma 3, it is easy to see that $\mathsf{ans}(C, G)$ is equivalent to the chase of Σ over G, for every RDF graph G.

For the other direction, we assume that no dependency in Σ has any variable in common. From each dependency d in Σ of form

$$\forall \bar{x}\ \forall \bar{y}\ (\varphi(\bar{x}, \bar{y}) \to \exists \bar{z}\ \psi(\bar{x}, \bar{z}))\,, \quad (4)$$

We create a c-query C_d as follows. Replace each variable z in ψ for a fresh blank node, and let T_ψ be a set of triples containing a triple (a, b, c) for each conjunct $OTriple(a, b, c)$ of ψ. Moreover, let P_ϕ be the pattern corresponding to ϕ, albeit without the existential quantification of the variables, as described in the proof of Lemma 5. Then note that from Lemma 5 and the definition of chase, the query $\mathsf{CONSTRUCT}\ T_\psi\mathsf{WHERE}\ P_\phi$ is actually equivalent to the mapping Σ_d containing only d.

Once again from Lemma 32 it is easy to see that the union of all graphs in $\{\mathsf{ans}(C_d, G) \mid d \in \Sigma\}$ corresponds to the chase of Σ over G, as we have shown that each C_d actually corresponds to the chase of d over G. Since none of these C_ds have variables in common, we can merge all of them into a single pattern without altering their semantics.

Proof of Proposition 19

From Proposition 18 and a simple observation of the chase algorithm, we know that the number of triples that contain a particular blank node in the result of a c-pattern C over a

graph G does not depend on the size of the graph, but rather on the amount of triples in the CONSTRUCT statement of C.

However, the following example shows construct patterns C_1 and C_2 and a family of RDF graphs $\{G_i \mid i > 1\}$ so that the number of triples connected to a blank node in $C_2(C_1(G_i))$ increases as we increase i. We should note that the example is heavily inspired in a classical result about composition of st-tgds in data exchange ([12]).

C-pattern C_1 is

```
CONSTRUCT{(?n, course, ?c), (?n, student, _:s)}
WHERE {(?n, takes, ?c)}
```

And C_2 is

```
CONSTRUCT{(?s, enrollment, ?c)}
WHERE{(?n, course, ?s) AND (?n, student, ?c)}
```

Intuitively, C_1 assigns, for each course $?n$ that the student takes, a new identifier for this student. Then, afterwards, C_2 takes the RDF graph constructed by C_1, and links each of the identifiers created for this student with each of the courses he's taking.

Thus, if one defines G_i as the RDF graph containing triples $\{(A, takes, B_1), \ldots, (A, takes, B_i)\}$, then $C_1(G_i)$ is the graph containing $\{(A, course, B_1), \ldots, (A, course, B_i)\} \cup \{(A, student, \_ : n_1), \ldots, (A, student, \_ : n_i)\}$. A simple inspection then reveals that each such blank $\_ : n_j$ forms part of triples $\{(B_1, enrollment, \_ : n_j), \ldots, (B_i, enrollment, \_ : n_j)\}$ in the graph $C_2(C_1(G_i))$.

Clearly, it is not possible to build a set of st-tgds Σ that when chasing over G_i produces precisely $C_2(C_1(G_i))$. The proof then follows from Proposition 18.

Proof of Proposition 21

Follows from Theorem 14 and an inspection of the proof of Proposition 18, because the constructed mapping is in this case an OPT-free mapping.

Proof of Proposition 26

For the proof it suffices to show the following.

▶ **Lemma 33.** *Let D and D' be two datasets that differ only on the graph named T, and if $\langle T, G\rangle$ belongs to D and $\langle T, G'\rangle$ belongs to D', then $T \subseteq T'$. Moreover, consider a semi-positive recursive query of form* WITH RECURSIVE T AS $\{q_1\}q_2$. *Then* $\mathsf{ans}((,q)_1, D) \subseteq \mathsf{ans}((,q)_1, D')$.

The proof of this lemma follows immediately from the proof of Lemma 5, since the semi-positiveness of q_1 guarantees that T appears only positively in the translation of q_1 over $\mathrm{FO}_{\mathsf{rdf}}$, and that no predicate $Named(a, b, c, d)$ appears negated if a is not a constant.

This establishes again that the operator that we are considering is monotone, and therefore a unique fixed point exists (c.f. [1]).

Proof of Theorem 27

Let $\Pi = \{\Pi_1, \ldots, \Pi_n\}$ be a program in $\mathrm{Datalog}^{\mathsf{rbr}}_{\mathsf{rdf}}$.

For each rule R of Π_i of form

$$P_i(\bar{x}_i) \leftarrow *R^i_1(\bar{z}^i_1), \ldots, *R^i_m(\bar{z}^i_m)$$

where each $*R_j(\bar{z}_j)$ is either P_ℓ, with $\ell \leq i$, or an EDB of the program, or the negation of a P_ℓ with $\ell < i$; let $T_1, \ldots, T_n$ be fresh IRIs, and let $\tau(R)$ be the result of replacing each IDB of form $P_\ell(\bar{z})$ for $Named(T_\ell, \bar{z})$ in both the body and head of P_i.

Construct a c-query Q_{Π_i} whose set of mappings corresponds to the union of query $\phi(\bar{x}_i) = *R^i_1(\bar{z}^i_1) \wedge \cdots \wedge *R^i_m(\bar{z}^i_m)$, according to Lemma 5, of all such $\tau(R)$, for each rule in Π_i.

Further, note that such query can be actually stated as query that satisfies the conditions of semi-positiveness, as it does not use predicates $Named(a, b, c, d)$ where a is a variable, and all instances of $Named(T_i, \bar{z})$ appear positive, meaning that negation shall not be needed for anything mentioning $Named(T_i, \bar{z})$.

By induction on the length of the proof it is immediate to show that the graph computed by the query

$$\begin{array}{l}\mathsf{WITH\,RECURSIVE}\, T_1\, \mathsf{AS}\{\mathsf{CONSTRUCT}(\bar{x}_1)\, \mathsf{FROM}\, Q_{P_1}\} \\ \quad \{\mathsf{WITH\,RECURSIVE}\, T_2\, \mathsf{AS}\{\mathsf{CONSTRUCT}(\bar{x}_1)\, \mathsf{FROM}\, Q_{P_1}\} \\ \qquad \{\cdots \{\mathsf{WITH\,RECURSIVE}\, T_n\, \mathsf{AS}\{Q_{P_n}\}\mathsf{CONSTRUCT}(\bar{x})\, \mathsf{FROM\,NAMED}\, T_n\} \cdots\}\}\end{array} \tag{5}$$

corresponds to the answer of the rule P_n.

For the other direction, consider a semi-positive recursive SPARQL of form 5. We are assuming that the objective is to output the constructed graph, but without loss of generality the following argument can be extended to a query of the above form where the final non recursive query is any c-query.

We then have that none of $Q_{P_1}, \ldots, Q_{P_n}$ is recursive, and that they satisfy the restrictions of semi-positiveness. From the proof of Lemma 5 we have that each of Q_{P_i} can be transformed into an FO query over $\mathrm{FO}_{\mathsf{rdf}}$ that is of the following form: no predicate $Named(a, b, c, d)$, where a is a variable or a T_i, appears under negation.

In order to convert this into a datalog program with rule-by-rule stratified negation, there are some technicalities:

First, every instance of $Named(a, b, c, d)$, where a is a variable, must be transformed into the disjunction of $Named(a, b, c, d) \wedge a \neq T_1 \wedge \cdots \wedge T_n$ and $Named(a, b, c, d) \wedge a = T_i$, for each T_i. Afterwards we can replace all instances of $Named(a, b, c, d) \wedge a = T_i$ or $Named(T_i, b, c, d)$ for a new predicate $T_i(b, c, d)$.

Now converting the resulting FO queries into Datalog (as done, for instance, in [1]), yields that their negation is stratified, because no $Named(T_i, b, c, d)$ or $Named(a, b, c, d)$, with a a variable, was under the scope of a negation in the previous query (this is guaranteed by the semi-positiveness). The union of each of the programs gives us a datalog program that has rule-by-rule stratified negation, and is equivalent to the original query.